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Type-IIB orientifolds on Gepner points

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Abstract: We study various aspects of orientifold projections of type IIB closed string theory on Gepner points in different dimensions. The open string sector is introduced, in the usual constructive way, in order to cancel RR charges carried by orientifold planes. Moddings by cyclic permutations of the internal $N = 2$ superconformal blocks as well as by discrete phase symmetries are implemented. Reduction in the number of generations, breaking or enhancements of gauge symmetries and topology changes are shown to be induced by such moddings. Antibranes sector is also considered; in particular we show how non supersymmetric models with antibranes and free of closed and open tachyons do appear in this context. A systematic study of consistent models in $D = 8$ dimensions and some illustrative examples in $D = 6$ and $D = 4$ dimensions are presented.

Keywords: D-branes, Superstring Vacua.

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1. Introduction

Compactifications of ten dimensional $E_8 \times E_8$ heterotic string down to four dimensional theories with close resemblance to the Standard Model (or extensions of it) defined the scenario for the so called string phenomenology, since the middle eighties. The guide lines were established in [1] where $E_8 \times E_8$ heterotic string compactified on a Calabi-Yau manifold was shown to lead to an $N = 1$ three generations E_6 model. Further breaking of E_6 may be achieved, for instance, by turning on Wilson lines. Quite soon, also other compactifications were considered according to these ideas, such as those involving compact orbifolds or free fermionic string models in $E_8 \times E_8$ and, in a less degree, SO(32) heterotic string [2]. For pioneer work on type-I model construction see [3] and references therein.

Particularly relevant for our following discussion are the Gepner models proposed in [4]. That work provides an algebraic construction of supersymmetric string theory in even D smaller than 10 dimensions, in terms of solvable rational conformal internal theories, without any reference to the original theory in ten dimensions. Some evidence for identification of such constructions with Calabi-Yau compactifications was also advanced.

Since the middle nineties, the irruption of *dualities* has marked a drastic change in our view of string theories, both from theoretical and phenomenological perspectives. D-branes play a prominent role in this new approach.

The fact that branes localize gauge interactions on their world volumes sets a new scenario for string phenomenology, where particle theories are confined into world branes. Since several features of such theories appear to depend upon just the local behavior of strings in the vicinity of D-branes (without even considering compactifications), the appealing possibility of a *bottom-up* [5] approach opens up. In this approach a local world type-II brane model (resembling the Standard Model) is built up in a first step and is successively embedded in a global consistent string model. It proves to be very powerful for toroidal like compactifying manifolds with either branes at orbifold like singularities or at angles [6], needed in order to achieve chirality (see also [7] for intersecting branes and Calabi-Yau).

The relevant type-I string theory on generic Calabi-Yau manifolds is much more cumbersome. In particular, the geometry of D-branes becomes fuzzy and a *bottom-up* like construction becomes somewhat out of control. However, we would like to stress that important steps towards the understanding of the algebraic and geometrical interpretation of D-branes in such generic cases, have been achieved [8, 9, 10]. This might, hopefully, lead to a model building *bottom-up* like procedure in a near future.

Beyond their phenomenological interest, type-I Calabi-Yau compactifications provide a fruitful arena to study type-I - heterotic dualities. As we know, this is an essential ingredient in the program to realize the nature of M-theory. In particular, it appears worth studying compactifications in different dimensions as a relevant step pointing to establish connections within the intricate web of string dualities.

The present article deals with the building of such kind of models in the special points of Calabi-Yau moduli space described by Gepner models. Previous work on this subject has been developed in recent years. Open descendants of Gepner models have been discussed

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in [11, 12]. D-branes [13, 14, 15] and orientifold planes in these models have been considered in [16, 17].

Type-I open plus closed unoriented superstring theory in ten dimensions can be constructed from type-IIB superstrings. The IIB theory is known to be invariant under the exchange of left and right moving sectors. When such a symmetry is modded out the resultant theory seems to be inconsistent. Such inconsistency manifests, for instance, through the appearance of unphysical tadpoles in string amplitudes and can be interpreted as an unbalanced charge under RR closed string fields carried by orientifold 9-planes [18]. Full consistency is recovered by adding an open string sector with open strings ending on D9-branes carrying opposite RR charge. It is in this sense that the open string sector appears as a twisted sector for the left-right exchange projection. The solution of tadpole cancellation conditions fix the Chan Paton gauge groups. The same scheme is valid in lower dimensions whenever left and right movers are coupled symmetrically.

In the present paper we follow these steps by starting with type-IIB theories where the internal sector is built up from Gepner models. Our main aim here is to develop a systematic procedure to handle such models. In particular, we show how moddings by phase symmetries or by cyclic permutations may be implemented in order to achieve partial control on the number of generations, the breaking or enhancement of gauge symmetries and supersymmetry, etc. The introduction of antibranes sectors is also discussed. Several examples in $D = 8, 6$ and 4 dimensions are constructed and presented for illustrative purposes. A more biased study towards phenomenologically interesting models or the finding of heterotic duals, for which the methods developed here should be helpful, is postponed for future work.

The paper is organized as follows. In sections 2 and 3, which contain brief reviews of the partition function in type-I superstring theory and of Gepner models respectively, the main ideas of the construction are developed and notation is set up. The vacuum amplitude in type-I theory at Gepner points is discussed in section 4 and it is illustrated through explicit examples in $D = 8, 6$ and 4 spacetime dimensions in sections 5, 6 and 7, where the matter content and gauge groups of Chan Paton factors leading to consistent theories are specified. Moddings by cyclic permutations and by discrete phase symmetries are considered in section 8. Introduction of antibranes is briefly considered in section 9. Section 10 offers a brief summary and outlook. In order to keep track of the essential aspects of the construction many details are relegated to appendices. Appendix A summarizes explicit expressions and properties of the characters of the N=2 superconformal minimal models and the modular transformation properties of supersymmetric characters of N=2 strings. In appendix B we list the spectrum of states contained in the relevant characters of the Gepner models constructed in the main body of the article.

2. Vacuum amplitude in type-I superstring

In this section we review the construction of type-I superstring theory (see the original work [26, 27] as well as the excellent review [28] and references therein).

Consider the type-IIB torus partition function in D dimensions (We leave details for reference [18] and explicit examples for next section). It is schematically defined as

$$\mathcal{Z}_T(\tau,\bar{\tau}) = \sum_{a,b} \chi_a(\tau)\mathcal{N}^{ab}\bar{\chi}_b(\bar{\tau}) \tag{2.1}$$

where the characters $\chi_a(\tau) = \mathrm{Tr}_{\mathcal{H}_a} q^{L_0-\frac{c}{24}}$, with $q = e^{2i\pi\tau}$, span a representation of the modular group of the torus generated by S: $\tau \to -\frac{1}{\tau}$ and T: $\tau \to \tau+1$ transformations. $\mathcal{H}_a$ is the Hilbert space of a conformal field theory with central charge $c = 15$ generated from a conformal primary state ϕ_a (similarly for the right moving algebra).

In particular $\chi_a(-\frac{1}{\tau}) = S_{aa'}\chi_{a'}(\tau)$ and modular invariance requires $S\mathcal{N}S^{-1} = \mathcal{N}$. Generically the characters can be split into a spacetime piece, contributing with $c_{st} = \bar{c}_{st} = \frac{3}{2}D$ and an internal sector with $c_{\rm int} = \bar{c}_{\rm int} = \frac{3}{2}(10-D)$. We are looking for left-right symmetric theories and therefore we must also require $\mathcal{N}^{ab} = \mathcal{N}^{ba}$.

Let Ω be the reversing order (orientifolding) operator permuting right and left movers. Modding by order reversal symmetry is then implemented by introducing the projection operator $\frac{1}{2}(1+\Omega)$ into the torus partition function. The resulting vacuum amplitude reads

$$\mathcal{Z}_\Omega(\tau,\bar{\tau}) = \mathcal{Z}_T(\tau,\bar{\tau}) + \mathcal{Z}_K(\tau-\bar{\tau})\,. \tag{2.2}$$

The first contribution is just the symmetrization (or anti-symmetrization in case states anticommute) of left and right sector contributions indicating that two states differing by a left-right ordering must be counted once. The second term is the Klein bottle contribution and takes into account states that are exactly the same in both sectors. In such case, the operator $e^{2i\pi\tau L_0}e^{-2i\pi\bar{\tau}\bar{L}_0}$, when acting on the same states, becomes $e^{2i\pi 2it_K L_0}$ with $\tau - \bar{\tau} = 2it_K$ and thus

$$\mathcal{Z}_K(2it_K) = \frac{1}{2}\sum_a \mathcal{K}^a \chi_a(2it_K)\,, \tag{2.3}$$

where $|\mathcal{K}^a| = \mathcal{N}^{aa}$ (there is a sign freedom in this definition which we fix by imposing consistency conditions [19, 20]). The Klein bottle amplitude in the *transverse channel* is obtained by performing an S modular transformation such that

$$\tilde{\mathcal{Z}}_K(il) = \frac{1}{2}\sum_a O_a^2 \chi_a(il) \tag{2.4}$$

with $l = 1/2t_K$ and

$$O_a^2 = 2^D \mathcal{K}^b S_{ba}\,. \tag{2.5}$$

This notation for the closed channel coefficients highlights the fact that the Klein bottle transverse channel represents a closed string propagating between two crosscaps (orientifold planes) which act like boundaries. This amplitude must still be integrated over the tube length. Since closed string states of mass m contribute as e^{-lm^2} in the character, one concludes that generically massless states will lead to tadpole like divergences in the limit $l \to \infty$. While NSNS massless states could presumably be interpreted as background redefinitions [21, 22], RR tadpoles lead, as mentioned, to unavoidable inconsistencies. Note that for such fields propagating in χ_a, O_a represents the charge of the orientifold plane (crosscap) under them.

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Inclusion of an open string sector with D-branes carrying $-O_a$ RR charge provides a way for having a consistent theory [26, 27] (see also [23, 24, 25, 28]) with net vanishing charge.[1]

An open string cylinder amplitude, representing strings propagating between two D-branes, and a Möbius strip amplitude with strings propagating between orientifold planes and D-branes must be included. In the long tube limit the sum of the contributions from the Klein bottle, cylinder and Möbius strip in the transverse channel must then factorize as

$$\tilde{\mathcal{Z}}_K(il) + \tilde{\mathcal{Z}}_M(il) + \tilde{\mathcal{Z}}_C(il) \to \sum_a (O_a + D_a)^2 \frac{1}{{m_a}^2} = \sum_a (O_a^2 + 2O_a D_a + D_a^2) \frac{1}{{m_a}^2} \qquad (2.6)$$

where m_a is the mass of the state in χ_a. For massless RR fields D_a is the D-brane RR charge and absence of divergences requires

$$O_a + D_a = 0\,. \qquad (2.7)$$

The generic form of the cylinder amplitude in the direct channel should read

$$\mathcal{Z}_C(it_C) = \frac{1}{2} \sum_a \mathcal{C}_a \chi_a(it_C)\,, \qquad (2.8)$$

where

$$\mathcal{C}_a = C_{jka} n_j n_k \qquad (2.9)$$

represents the multiplicity of states contained in $\chi_a(it)$ and n_j, n_k are Chan-Paton multiplicities. n_j can be interpreted as the number of branes of type j where the string endpoints must be attached.[2] $\sum_j n_j = N_B$ is the total number of D-branes. Let us mention that, in general, while the index a runs over the different conformal highest weight representations defining the characters χ_a, j indices are not necessarily in a one to one correspondence with them. However, there is such correspondence for charge conjugation modular invariants [34, 26, 27].

C_{ija} must thus be positive integers. Actually, as we discuss below, $C_{ija} = 0, 1, 2$. The transverse channel representation of this amplitude reads

$$\tilde{\mathcal{Z}}_C(il) = \frac{1}{2} \sum_a D_a^2 \chi_a(il) \qquad (2.10)$$

with $D_a = D_{ja} n_j$ and

$$(D_{ja} n_j)^2 = \mathcal{C}_b S_{ba} = C_{jkb} n_j n_k S_{ba}\,. \qquad (2.11)$$

The Möbius strip amplitude presents some additional subtleties since the modulus $it_M + 1/2$ is not purely imaginary. Indeed the characters are given by

$$\chi_a^{\Omega}(it_M) \equiv \mathrm{Tr}_{\mathcal{H}_a}(e^{\pi i t (L_0 - \frac{c}{24})}\Omega) = \chi_a\left(it_M + \frac{1}{2}\right), \qquad (2.12)$$

[1] In section 8 we consider the possibility of including antibranes with the consequent breaking of supersymmetry.

[2] j should, presumably, have a topological interpretation, as it is the case for branes at orbifold singularities, where it labels the monodromy at the singularity.

and this introduces relative signs for the oscillator excitations at the various mass levels leading to complex characters. The amplitude in the direct channel takes the form

$$\mathcal{Z}_M(it_M) = \frac{1}{2}\sum_a \mathcal{M}_a \hat{\chi}_a\left(it_M + \frac{1}{2}\right) \tag{2.13}$$

where now

$$\mathcal{M}_a = M_{ja} n_j \tag{2.14}$$

are integer numbers and the "hat" in the characters indicates that the phase $e^{i\pi(h-c/24)}$ has been extracted to make them real.

The characters in the direct and transverse channels of the Möbius strip are related by the transformation [26] P: $it_M + 1/2 \to i/4t_M + 1/2$. This can be generated from the modular transformations S and T as $\mathsf{P} = \mathsf{TST}^2\mathsf{S}$. The transverse channel representing a closed string propagating between a D-brane and an orientifold plane must read

$$\tilde{\mathcal{Z}}_M(il) = \frac{1}{2}\sum_a O_a(D_{ja} n_j)\hat{\chi}_a(il + \frac{1}{2}) \tag{2.15}$$

with

$$O_a(D_{ja} n_j) = 2^{\frac{D}{2}} \mathcal{M}_b P_{ba} = 2^{\frac{D}{2}} M_{jb} n_j P_{ba} \tag{2.16}$$

Notice that the tube length $l = 1/2t_K = 1/t_C = 1/4t_M = -\frac{1}{2\pi}\ln q$ for the different string amplitudes must be the same in order for them to be comparable. We thus see that in the long length limit ($\hat{\chi}_a(il + \frac{1}{2}) \to \chi_a(il)$) the correct closed string channel factorization (2.6) is obtained.

In forthcoming sections we will apply this open descendant construction to IIB theories where the internal sector is built up from Gepner models.

Before closing this section let us make a few comments about the open string spectrum. Notice that the coefficients of $q^{m_a^2}$ in a q expansion of cylinder + Möbius strip direct channel amplitudes, which are proportional to

$$\frac{1}{2}[(C_{jka} n_j n_k) \pm (M_{ja} n_j)]\,, \tag{2.17}$$

are nothing but the multiplicities of open string fields of mass m_a. In principle, such multiplicities and the spacetime transformation properties of the corresponding characters should allow us to reconstruct the spectrum. Notice that even and odd levels in the Möbius strip differ in sign, due to the $1/2$ term in the argument of the character.

Since open string gauge group representations are generated by the Chan Paton indices in the two string endpoints, we can infer that only symplectic, orthogonal and/or unitary groups are allowed [18, 28]. Moreover, only adjoint (**Adj**), symmetric ($\square\!\square$) or antisymmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) (and their conjugate) representations can be built from Chan-Paton factors ending on the same type of brane. The quadratic part of such representations comes from cylinder contributions and therefore we must expect that $C_{iia} = 0, 1, 2$. Recall also that, had we obtained a symmetric (antisymmetric) representation at some mass level then the next level would contain an antisymmetric (symmetric) one and so on, due to the alternate signs in the Möbius strip contribution.

If two gauge groups $G_i \times G_j$ were present, bi-fundamental $[(\square_i, \overline{\square}_j)]$ representations corresponding to endpoints ending on two different sets of n_i and n_j D-branes respectively could also arise and thus $C_{jia} = 0, 1$. Linear terms in n_j in (2.17) coming from the Möbius strip amplitude must complete the two index representations and thus $M_{ja} = 0, 1, -1$.

Moreover, $C_{iia} = 2$ for the character χ_a, with the corresponding null Möbius strip coefficient $M_{ia} = 0$, would indicate a $\mathrm{U}(n_i)$ adjoint representation.[3] Similarly if $C_{jia} = 1$ with $i \neq j$, then $M_{ja} = M_{ia} = 0$.

Once the Klein bottle partition function is obtained from the left-right symmetric type-IIB torus partition function, our construction of the open string sector will completely rely on

1. Factorization
2. Massless RR tadpole cancellation
3. Consistency restrictions on the integer coefficients C_{jia} and M_{ja}.

3. Review of Gepner models

Gepner has shown how to construct supersymmetric closed string theories in four spacetime dimensions replacing the geometrical notion of curling up the extra dimensions into a compact internal manifold by an algebraic procedure where the internal sector consists of tensor products of N=2 superconformal minimal models with total central charge $c_{\rm int} = 9$ [4]. Spacetime supersymmetry and modular invariance are implemented by keeping in the spectrum only states for which the total U(1) charge is an odd integer. Let us briefly review Gepner's construction to set up notation.

A consistent string theory in D spacetime dimensions requires an internal conformal field theory with $c_{\rm int} = 12 - \frac{3}{2}(D-2)$ in the light cone gauge. N=1 spacetime supersymmetry is achieved if the internal CFT has N=2 supersymmetry. The $(D-2)$ spacetime bosons and fermions X^μ, ψ^μ define a CFT with $c_{st} = \frac{3}{2}(D-2)$ and they realize an N=2 superconformal algebra for even D.

Gepner models represent an explicit algebraic construction of supersymmetric string vacua where the internal sector is given by a tensor product of r copies of N=2 superconformal minimal models with levels k_j, $j = 1, \ldots, r$ and central charge

$$c = \frac{3k}{k+2} \qquad k = 1, 2, \ldots\,. \tag{3.1}$$

$N = 2$ Superconformal minimal models. Let us recall the N=2 superconformal algebra here for completeness, namely

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
[L_m, J_n] &= -nJ_{m+n}
\end{aligned}$$

[3] Actually, once a unitary group is identified, it proves useful to rewrite the term n^2 as $n\bar{n}$ (see for instance [3]). Even if numerically $n = \bar{n}$, this allows us to distinguish complex representations.

$$
\begin{aligned}
[L_m, G_r^\pm] &= \left(\frac{m}{2} - r\right) G_{m+r}^\pm \\
[J_m, J_n] &= \frac{nc}{3}\delta_{n+m,0} \\
[J_m, G_r^\pm] &= \pm G_{m+r}^\pm \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \\
\{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0\,.
\end{aligned}
\tag{3.2}
$$

Integer or semi-integer modding r, s correspond to the R or NS sector respectively. The spectral flow symmetry of the algebra allows to consider twisted sectors interpolating between R and NS. In fact the following operators

$$
\begin{aligned}
\tilde{L}_m &= L_m + \frac{n}{2}J_m + \frac{c}{6}n^2\delta_{m,0} \\
\tilde{G}_r^\pm &= G_{r\pm\frac{n}{2}}^\pm \\
\tilde{J}_m &= J_m + \frac{c}{6}n\delta_{m,0}\,,
\end{aligned}
\tag{3.3}
$$

generate an isomorphic N=2 algebra with the same central charge and modified $G_r^\pm \to G_{r\pm\frac{n}{2}}^\pm$ modding.

As is well known unitary representations of the N=2 superconformal algebra are found for discrete values of the central charge. For $c < 3$ the discrete minimal series is given by (3.1). The primary fields of the minimal models are labelled by three integers (l, q, s) such that $l = 0, 1, \ldots, k$; $l + q + s = 0$ mod 2 and they belong to the NS or R sector when $l + q$ is even or odd respectively. The conformal dimensions and charges of the highest weight states are given by

$$
\Delta_{l,q,s} = \frac{l(l+2) - q^2}{4(k+2)} + \frac{s^2}{8} \mod 1 \tag{3.4}
$$

$$
Q_{l,q,s} = -\frac{q}{k+2} + \frac{s}{2} \mod 2\,. \tag{3.5}
$$

Two representations labelled by (l', q', s') and (l, q, s) are equivalent, *i.e.* they correspond to the same state, if

$$
l' = l\,, \qquad q' = q \bmod 2(k+2)\,, \qquad s' = s \bmod 4 \tag{3.6}
$$

or

$$
l' = k - l\,, \qquad q' = q + k + 2\,, \qquad s' = s + 2\,. \tag{3.7}
$$

The exact conformal dimension and charge of the highest weight state in the representation (l, q, s) are obtained from equations (3.4) and (3.5) using the identifications above to bring (l, q, s) to the *standard range* given by

$$
l = 0, 1, \ldots, k\,; \qquad |q - s| \le l\,; \qquad l + q + s = 0 \bmod 2 \tag{3.8}
$$

and $|s|$ is the minimum value among those in (3.6) and (3.7).

The primary fields obey the inequalities

$$\Delta_{l,q,s} \geq \frac{|Q_{l,q,s}|}{2} \tag{3.9}$$

for representations with even s which belong to the NS sector, while for odd s, which belong to the R sector, they satisfy

$$\Delta_{l,q,s} \geq \frac{c}{24} \,. \tag{3.10}$$

The twisting operation (3.3) corresponds to applying n times the transformations $q \to q+1,\ s \to s+1$. The conformal dimensions and charges of the fields in the n-th twisted sector, when both l, q, s and $l, q+n, s+n$ are in the standard range, are obtained as

$$\begin{aligned} \Delta_{l,q+n,s+n} &= \Delta^n_{l,q,s} = \Delta^0_{l,q,s} + \frac{n}{2} Q^0_{l,q,s} + n^2 \frac{c}{24} \\ Q_{l,q+n,s+n} &= Q^n_{l,q,s} = Q^0_{l,q,s} + n\frac{c}{6} \,. \end{aligned} \tag{3.11}$$

Notice that even n interpolates between the same sector whereas odd n exchanges R and NS sectors. When the transformations $q \to q+n, s \to s+n$ take them outside the standard range the expressions for the conformal dimension and charge differ from (3.11) by an integer and an even number respectively. Moreover the identifications (3.6) imply that one comes back to the original representation after twisting by $n = 2(k+2)$ for even k and by $n = 4(k+2)$ for odd k. (A particular case is given by $l = k/2$ for even k where the identification (3.7) implies that the original representation is re-obtained after twisting by $n = k+2$).

The partition function of the minimal models on the torus can be written in terms of the characters of the irreducible representations as

$$\mathcal{Z}^{(m.m.)}_T(\tau) = \sum_{(l,q,s),(\bar{l},\bar{q},\bar{s})} \mathcal{N}_{(l,q,s),(\bar{l},\bar{q},\bar{s})} \chi_{(l,q,s)}(\tau,0) \chi^*_{(\bar{l},\bar{q},\bar{s})}(\bar{\tau},0) \tag{3.12}$$

where the coefficients $\mathcal{N}_{(l,q,s),(\bar{l},\bar{q},\bar{s})}$ are non negative integer numbers which count the number of times the irreducible representation $(l,q,s)\otimes(\bar{l},\bar{q},\bar{s})$ is contained in $\mathcal{H}$. The existence of a unique ground state requires $\mathcal{N}_{(0,0,0),(0,0,0)} = 1$. The characters in the sector $\mathcal{H}_{(l,q,s)}$ are given by

$$\chi_{(l,q,s)}(\tau,z) = \mathrm{Tr}_{\mathcal{H}_{(l,q,s)}} \left(e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{2\pi i z J_0} \right) \tag{3.13}$$

in the holomorphic untwisted sector, and by

$$\chi_{(l,q+n,s+n)}(\tau,z) \equiv \mathbf{Q}^n \chi_{(l,q,s)}(\tau,z) \tag{3.14}$$

in the holomorphic sector twisted by n, where $\mathbf{Q}$ is the operator interpolating between NS and R sectors, and similarly for the antiholomorphic part.

The Hilbert space can be decomposed into two subspaces $\mathcal{H}_{(l,q)} = \mathcal{H}_{(l,q,s)} + \mathcal{H}_{(l,q,s+2)}$, where each subspace is generated by an even number of $G^\pm_r$ on the primary fields $|\Psi_{(l,q,s)}>$ and $|\Psi_{(l,q,s+2)}>$. Both subspaces are related by the action of one $G^\pm$. Therefore it is convenient to define

$$\chi_{l,q}(\tau,z) \equiv \mathrm{Tr}_{\mathcal{H}_{l,q}} (e^{2\pi i (L_0 - \frac{c}{24})\tau} e^{2\pi i J_0 z}) = \chi_{(l,q,s)}(\tau,z) + \chi_{(l,q,s+2)}(\tau,z) \,. \tag{3.15}$$

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Explicit expressions and properties of the characters of the $N = 2$ superconformal minimal models are included in appendix A.

The character $\chi_{l,q}$ with even $l+q$ contains two families of states with primaries (l,q,s) and $(l,q,s+2)$ whose conformal dimensions differ by 1/2. It is always possible to choose $s = 0$ for the primary with the smaller conformal dimension and (l,q,s) in the standard range. This state has conformal dimension and charge given by (3.4) and (3.5), namely

$$\begin{aligned} \Delta_{l,q} &= \frac{l(l+2)-q^2}{4m} - \frac{c}{24} \\ Q_{l,q} &= -\frac{q}{m} \quad (|q| \leq l, l+q = even) \end{aligned} \tag{3.16}$$

where $\Delta_{l,q} \equiv \Delta_{l,q,0}$ and $Q_{l,q} \equiv Q_{l,q,0}$. For the other primary, $|s| \geq 2$.

In the R sector the two highest weight states have the same conformal dimension and their charges differ by one, except when one of the conformal dimensions is $c/24$. It is always possible to choose $s = -1$ for the representation in which the highest weight state has smaller charge. Thus the conformal dimension and charge of this state are given by

$$\begin{aligned} &\Delta_{l,q} = \frac{l(l+2)-q^2}{4(k+2)} + \frac{1}{8}\,; \quad Q_{l,q} = -\frac{q}{k+2} - \frac{1}{2} \\ &l+q = odd\,, \quad -l-1 \leq q \leq l-1\,. \end{aligned} \tag{3.17}$$

For $q \neq -l-1$ ($\Delta_{l,q} > c/24$) the highest weight state $(l,q,s+2)$ can be obtained choosing $s = 1$; it has the same conformal dimension as (l,q,s) but its charge is $Q_{l,q}$ +1 whereas for $q = -l-1$ ($\Delta_{l,q} = \frac{c}{24}$), the conformal dimension is $\Delta_{l,q} + 1$.

From the equivalence relations (3.6) and (3.7) one can show the following identities

$$\chi_{l,q}(\tau,z) = \chi_{k-l,q+k+2}(\tau,z) = \chi_{l,q+2(k+2)}(\tau,z)\,. \tag{3.18}$$

Charge conjugate characters contain highest weight states with charges $Q_{l,q}$ and $Q_{l,-q}$ and they satisfy the following equality

$$\chi_{l,q}(\tau,z) = \chi_{l,-q}(\tau,-z)\,. \tag{3.19}$$

Modular transformations map one sector into another and thus modular invariance requires considering the four sectors NS$^{\pm}$, R$^{\pm}$, defined as

$$\chi^{NS^{\pm}}_{l,q} = \frac{1}{2}(\chi_{(l,q,0)} \pm \chi_{(l,q,2)}) \tag{3.20}$$

$$\chi^{R^{\pm}}_{l,q} = \frac{1}{2}(\chi_{(l,q-1,-1)} \pm \chi_{(l,q-1,1)})\,. \tag{3.21}$$

Note that $\chi^{NS^+}_{l,q} = \chi_{l,q}$. They can all be written in terms of $\chi^{NS^+}_{l,q}$ by shifting z as

$$\begin{aligned} \chi^{NS^-}_{l,q}(\tau,z) &= e^{-\pi i Q}\chi^{NS^+}_{l,q}\left(\tau, z+\frac{1}{2}\right) \\ \chi^{R^+}_{l,q}(\tau,z) &= e^{2\pi i\tau c/24}e^{2\pi i z c/6}\chi^{NS^+}_{l,q-1}\left(\tau, z+\frac{1}{2}\tau\right) \\ \chi^{NS^-}_{l,q}(\tau,z) &= e^{-\pi i Q}e^{2\pi i\tau c/24}e^{2\pi i z c/6}\chi^{NS^+}_{l,q-1}\left(\tau, z+\frac{1}{2}\tau+\frac{1}{2}\right)\,. \end{aligned} \tag{3.22}$$

Under S: $\tau \to -\frac{1}{\tau}$, the minimal characters $\chi_{l,q}$ with even $l+q$ transform as

$$\chi_{l,q}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{2\pi i \frac{z^2 c}{6\tau}} \sum_{l',q'} S_{l,q;l',q'} \chi_{l',q'}(\tau,z)\,, \tag{3.23}$$

or equivalently as

$$\chi_{l,q}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{2\pi i \frac{z^2 c}{6\tau}} \sum_{l',q'} S^{-1}_{l,q;l',q'} \chi_{l',q'}(\tau,-z) \tag{3.24}$$

where the S matrix is given by

$$S_{l,q;l',q'} \equiv \frac{2}{(k+2)} e^{\pi i \frac{qq'}{k+2}} \sin\left(\pi \frac{(l+1)(l'+1)}{k+2}\right) \tag{3.25}$$

(for even $(l+q)$). It verifies the following equality

$$S_{l,q;l',q'} = S_{k-l,q+k+2;l',q'}\,. \tag{3.26}$$

Applying the S transformation twice one gets

$$\chi_{l,q}(\tau,z) = \sum_{l',q'} S^2_{l,q;l',q'} \chi_{l',q'}(\tau,-z) \tag{3.27}$$

which leads to the *charge conjugation* matrix

$$S^2 = C\,, \qquad C_{l,q;l',q'} = \delta_{l,q;l,-q}\,.$$

Using the explicit expressions for the characters it is easy to verify the following action of the S transformation: $\mathrm{NS}^+ \to \mathrm{NS}^+$, $\mathrm{NS}^- \leftrightarrow \mathrm{R}^+$, $\mathrm{R}^- \to \mathrm{R}^-$.

Under $\mathsf{T}: \tau \to \tau+1$ the characters transform as

$$\begin{aligned}
\chi^{NS^+}_{l,q}(\tau+1,z) &= e^{2\pi i(\Delta_{l,q}-\frac{c}{24})}\chi_{l,q,0}(\tau,z) + e^{2\pi i(\Delta_{l,q}+\frac{odd}{2}-\frac{c}{24})}\chi_{l,q,2}\left(\tau,z+\frac{1}{2}\right)\\
&= e^{2\pi i(\Delta_{l,q}-\frac{c}{24})}\chi^{NS^-}_{l,q}(\tau,z)\\
\chi^{NS^-}_{l,q}(\tau+1,z) &= e^{2\pi i(\Delta_{l,q}-\frac{c}{24})}\chi^{NS^+}_{l,q}(\tau,z)\\
\chi^{R^\pm}_{l,q}(\tau+1,z) &= e^{2\pi i(\Delta_{l,q-1,-1}-\frac{c}{24})}\chi^{R^\pm}_{l,q}(\tau,z) = e^{2\pi i(\Delta_{l,q}-\frac{Q_{l,q}}{2})}\chi^{R^\pm}_{l,q}(\tau,z)
\end{aligned} \tag{3.28}$$

Therefore T: $\mathrm{NS}^\pm \to \mathrm{NS}^\mp$; $\mathrm{R}^\pm \to \mathrm{R}^\pm$.

N=2 strings. A D dimensional string theory is obtained by taking the tensor product of r internal N=2 SCFTs such that $\sum_{i=1}^r c_i = 12 - \frac{3}{2}(D-2)$ and appending the spacetime contribution. Let us start by reviewing the spacetime part.

The $(D-2)$ spacetime bosons and fermions realize a (2,2) superconformal algebra. The fermions $\psi^\mu(z)$ $(\mu = 1,\ldots,D-1)$ exhibit a SO$(D-2)$ symmetry which require the states to be in unitary representations of the affine transverse Lorentz algebra at level $k=1$. These are the scalar, vector, spinor and conjugate spinor representations labelled respectively by $\lambda = 0, 2, 1, -1$. The contribution of each pair of transverse dimensions to the spacetime characters is given by

$$\begin{aligned}
\Upsilon_{0(2)}(\tau, z) &= \frac{1}{2\eta(\tau)^3}\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau, z) \pm \vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(\tau, z)\right) \\
\Upsilon_{1(-1)}(\tau, z) &= \frac{1}{2\eta(\tau)^3}\left(\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(\tau, z) \mp i\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(\tau, z)\right).
\end{aligned} \tag{3.29}$$

where the upper (lower) sign corresponds to first (second) subindex in the character. The conformal spacetime dimensions and charges of the states are

$$\Delta_{st} = \frac{\lambda^2}{8}, \qquad Q_{st} = \frac{\lambda}{2}, \tag{3.30}$$

and the field identification is $\lambda' = \lambda \mod 4$.

Similarly as in (3.20) and (3.21) we define for each pair of transverse dimensions

$$\chi^{NS^\pm}(\tau, z) = \Upsilon_0(\tau, z) \pm \Upsilon_2(\tau, z); \qquad \chi^{R^\pm}(\tau, z) = \Upsilon_{-1}(\tau, z) \pm \Upsilon_1(\tau, z), \tag{3.31}$$

and it can be seen from (3.29) that $\chi^{R^-}(\tau, 0) \equiv 0$. We denote the spacetime characters in the ν-th twisted sector as

$$\chi_\nu(\tau, z) = \Upsilon_{0+\nu}(\tau, z) + \Upsilon_{2+\nu}(\tau, z). \tag{3.32}$$

Note that $\chi_\nu(\tau, z) = \chi^{NS^+}(\tau, z)$ if ν is even and $\chi_\nu(\tau, z) = \chi^{R^+}(\tau, z)$ if ν is odd.

The S modular transformation on these spacetime characters reduces to

$$\chi^{NS^+}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{e^{2\pi i \frac{z^2 c_{st}}{6\tau}}}{-i\tau} S_{st} \chi^{NS^+}(\tau, z),$$

where $c_{st} = \frac{3}{2}(D-2)$ and $S_{st} = 1$.

Putting everything together the character associated to a primary state of the full theory is given by the product of the contributions of spacetime times the r internal theories. In order to achieve N=1 supersymmetry on the world sheet all the states in the product must belong to a definite sector, *i.e.* NS (R) states must be tensored only with NS (R) states. Modular invariance requires odd total U(1) charge Q. These conditions lead to the following character

$$\chi_{\vec{\alpha}}(\tau, z) \equiv \hat{\mathcal{P}}_{GSO}\{\chi'_{\vec{\alpha}}(\tau, z)\} \equiv \hat{\mathcal{P}}_{GSO}\{[\chi_\nu(\tau, z)]^d \prod_{i=d+1}^{d+r} \chi_{\alpha_i}(\tau, z)\} \tag{3.33}$$

where $[\chi_\nu(\tau, z)]^d$ is the D dimensional spacetime character with $d = \frac{(D-2)}{2}$. Here $\vec{\alpha}$ is a $(d+r)$-component vector with entries $\alpha_i = \nu$ for $i = 1, \ldots, d$ and $\alpha_i = (l_i, q_i)$ for $i = d+1, \ldots, d+r$ denoting the full primary state of the product of internal and spacetime theories such that both $l_i + q_i$ and ν are even or odd. $\hat{\mathcal{P}}_{GSO}$ denotes the generalized GSO projection over states with odd U(1) charge.

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The action of the supersymmetry operator on the product theory can be expressed more conveniently introducing a vector $\vec{\alpha}^{(n)}$ with components $\alpha_i^{(n)} = \nu + n$ for $i = 1, \ldots, d$ and $\alpha_i^{(n)} = (l_i, q_i + n)$ for $i = d+1, \ldots, d+r$ as

$$\mathbf{Q}^n \chi_{\vec{\alpha}}(\tau, z) = \chi_{\vec{\alpha}^{(n)}}(\tau, z) \,. \tag{3.34}$$

Notice that if α_i denotes a state in the NS sector then $\alpha_i^{(n)}$ and therefore $\vec{\alpha}$ correspond to the NS sector.

$N = 1$ supersymmetry in spacetime requires summing over all twisted sectors. The identifications among characters allow to sum over n mod $2m$ where m is the *l.c.m.* of all the k_i+2 in the product. The *supersymmetric* character is finally given by (see appendix A)

$$\chi^{\rm susy}_{\vec{\alpha}}(\tau, z) = \sum_{n=0}^{2m-1} (-1)^n \chi_{\vec{\alpha}^{(n)}}(\tau, z) = \tag{3.35}$$
$$= \frac{1}{2m} \sum_{n,p \text{ mod } 2\text{m}} (-1)^{n+p} e^{2\pi i (n^2 \frac{c}{24}\tau + n\frac{c}{6} z)} \left[\chi_0\Big(\tau, z + \frac{n}{2}\tau + \frac{p}{2}\Big)\right]^d \prod_{i=1}^{r} \chi_{l_i, q_i}\Big(\tau, z + \frac{n}{2}\tau + \frac{p}{2}\Big)$$

with $c = 12$. NS or R sectors are obtained when summing over even or odd n, respectively and periodic $(+)$ or antiperiodic $(-)$ characters arise when summing over even or odd p, respectively.

This is a useful result. The open sector will be easily written down, in an explicit supersymmetric expression, as linear combinations of these characters.

Let us discuss some properties of the states contained in $\chi^{\rm susy}_{\vec{\alpha}}(\tau)$. It is convenient to introduce the following notation: $\vec{\beta}$ is a $d + r$ component vector with entries $\beta_i = \lambda_i$ for $i = 1, \ldots, d$ and $\beta_i = (l_i, q_i, s_i)$ for $i = d+1, \ldots, d+r$ and such that $l_i + q_i + s_i$ and λ_i are both even or odd. Analogously we can define the vector $\vec{\beta}^{(n)}$ obtained as $\lambda_i, q_i, s_i \to \lambda_i + n, q_i + n, s_i + n$. Due to the generalized GSO projection the states contained in the characters $\chi^{NS}_{\vec{\alpha}}$ and $\chi^{R}_{\vec{\alpha}}$ carry an index $\vec{\beta}$ such that $Q_{\vec{\beta}^{(n)}}$ is odd ($Q_{\vec{\beta}} = \sum_{i=1}^{d+r} Q_{\beta_i}$). Before GSO projecting, the charges of the states contained in the character are all related by

$$Q_{\vec{\beta}^{(n)}} = Q_{\vec{\beta}} + 2n \mod 2 \tag{3.36}$$

and therefore all the GSO projected products of states in $\chi_{\vec{\alpha}^{(n)}}$ for a given n can be obtained by twisting the GSO projected product of states for $n = 0$.

The full conformal dimensions of the fields in the n-th twisted sector are given by

$$\Delta_{\vec{\beta}^{(n)}} = \Delta_{\vec{\beta}} + \frac{n}{2} Q_{\vec{\beta}} + \frac{n^2}{2} \mod 1 \tag{3.37}$$

where $\Delta_{\vec{\beta}} = \sum_{i=1}^{d+r} \Delta_{\beta_i}$. Notice that the sum of the conformal dimensions of the untwisted states after GSO projecting differs by an integer from the sum of the conformal dimensions of the states twisted by n (since $\frac{n}{2}(Q_{\vec{\beta}} + n)$ is integer for odd $Q_{\vec{\beta}}$). Therefore all the states obtained by twisting an odd U(1) charge state have conformal dimension differing by an integer from that of the untwisted state.

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Let us now compare GSO projected states in the NS sector labelled with vectors $\vec{\beta}$ and $\vec{\beta}'$ having $l_i = l'_i, q_i = q'_i$, $\lambda_i \neq \lambda'_i$ and $s_i \neq s'_i$. The difference in their conformal dimensions and charges is given by the number of states with $s_i \neq 0$. Indeed considering that

$$\Delta_{\lambda+2} - \Delta_\lambda = \frac{1+\nu}{2} \mod 1\,; \quad Q_{\lambda+2} - Q_\lambda = 1 \mod 2$$
$$\Delta_{l,q,s+2} - \Delta_{l,q,s} = \frac{1+l+q+s}{2} \mod 1\,; \quad Q_{l,q,s+2} - Q_{l,q,s} = 1 \mod 2$$

it is easy to see that the states with odd U(1) charge verify

$$\Delta_{\vec{\beta}'} - \Delta_{\vec{\beta}} \in Z\,; \quad Q_{\vec{\beta}'} - Q_{\vec{\beta}} = 0 \mod 2 \tag{3.38}$$

We conclude that all the states contained in a given $\chi^{\rm susy}_{\vec{\alpha}}$ have conformal dimensions given by $\Delta_{\vec{\beta}^{(n)}} - 1/2 = \Delta_{\vec{\beta}_0} - \frac{Q_{\vec{\beta}_0}}{2} \mod 1$ ($\vec{\beta}_0$ is the vector $\vec{\beta}$ with $s_i = 0$ for all i). Since $\Delta_{\vec{\beta}_0} = \Delta_{\vec{\alpha}}$ and $Q_{\vec{\beta}_0} = Q_{\vec{\alpha}}$, finally

$$\Delta_{\vec{\beta}^{(n)}} - \frac{1}{2} = \Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2} \mod 1 \tag{3.39}$$

where $\vec{\alpha} \in$ NS. This is an important relation because it gives the conformal dimensions of the product of GSO projected states (modulo an integer number) from the conformal dimensions and charges of the highest weight states in the non-projected character $\prod_i \chi_{\alpha_i}$.

Taking into account that $\chi^{\rm susy}_{\vec{\alpha}}$ contains the sum over all twisted sectors and that all $\chi^{\rm susy}_{\vec{\alpha}^{(n)}}$ with even n contain the same representations, the following identities hold

$$\chi^{R/NS}_{\vec{\alpha}^{(n)}}(\tau, z) = \chi^{R/NS}_{\vec{\alpha}}(\tau, z)\,; \quad \chi^{\pm}_{\vec{\alpha}^{(n)}}(\tau, z) = \chi^{\pm}_{\vec{\alpha}}(\tau, z)\,. \tag{3.40}$$

One may thus choose a representative $\vec{\alpha}$ for all the equivalent vectors under twisting and the number of independent characters is then reduced by m for each R or NS sector. There is an important exception when some of the k_i are even: if $\vec{\alpha}$ contains $l_i = k_i/2$ for all even k_i then the number of supersymmetric characters related by twisting is $m/2$ and the states with $l_i = k_i/2$ are obtained twice in the sum over n from 0 to $2m-1$. We shall refer to these as *short vectors*.

The following relation between supersymmetric characters follows from the identity (3.19) between charge conjugate characters for each minimal model

$$\chi^{NS/R}_{\vec{\alpha}}(\tau, z) = \chi^{NS/R}_{\overline{(\vec{\alpha})}}(\tau, -z) \tag{3.41}$$

where $\overline{(\vec{\alpha})}$ is the vector obtained replacing q_i by $-q_i$ in $\vec{\alpha}$.

Modular transformations of the supersymmetric characters are discussed in appendix A.

4. Type-I superstring at Gepner points

The spectrum of perturbative type-II closed string states in Gepner models is contained in the full supersymmetric and modular invariant partition function for N=2 strings on the torus which is obtained combining the right and left sectors as

$$\mathcal{Z}_T(\tau,\bar\tau) = \sum_{\vec\alpha;\vec{\bar\alpha}} \mathcal{N}_{\vec\alpha;\vec{\bar\alpha}} \chi^{\rm susy}_{\vec\alpha}(\tau,0)\chi^{susy\ *}_{\vec{\bar\alpha}}(\bar\tau,0)\,, \tag{4.1}$$

and integrating over τ with the appropriate measure

$$\mathsf{Z}_T = \int \frac{d\tau d\bar\tau}{({\rm Im}\tau)^2}\mathcal{Z}_T(\tau,\bar\tau)\,. \tag{4.2}$$

Here $\mathcal{N}_{\vec\alpha,\vec{\bar\alpha}}$ are positive integer coefficients obtained from the product $\prod_{i=1}^r \mathcal{N}_{\alpha_i;\bar\alpha_i}$ of the individual minimal models such that the partition function is modular invariant.

In the following section we construct the partition functions for the unoriented and open descendants of type-IIB N=2 superstrings, *i.e.* the vacuum amplitudes from the Klein bottle, the Möbius strip and the cylinder, with special attention to the possible contributions to tadpoles and their cancellation.

4.1 Klein Bottle Amplitude

The partition function from the Klein bottle can be obtained from that of the torus as discussed in section 2. Integrating over t with the appropriate measure, the vacuum amplitude for the Klein bottle in the direct (closed) channel is given by

$$\begin{aligned}\mathsf{Z}_K &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}{\rm Tr}_{\mathcal{H}_{cl}}\left\{\Omega\ \exp\left[2\pi i(\tau(L_0-\frac{c}{24})-\bar\tau(\bar L_0-\frac{c}{24}))\right]\right\}\\ &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}\,(\frac{1}{4\pi^2\alpha' t})^{\frac{D}{2}}{\rm Tr}'_{\mathcal{H}_{cl}}\left\{\exp\left[-4\pi t(L_0-\frac{c}{24})\right]\right\}\end{aligned} \tag{4.3}$$

where ${\rm Tr}'$ denotes the trace over the discrete oscillator modes and the factor $(4\pi^2\alpha' t)^{-D/2}$ comes from the integral over the bosonic zero modes. The trace can be written in terms of the supersymmetric characters $\chi^{\rm susy}_{\vec\alpha}$ as

$$\mathcal{Z}_K(it) = \frac{1}{2}{\rm Tr}'_{\mathcal{H}_{cl}}\left\{\exp\left[-4\pi t(L_0-\frac{c}{24})\right]\right\} = \frac{1}{2}\sum_{\vec\alpha}\mathcal{K}_{\vec\alpha}\chi^{\rm susy}_{\vec\alpha}(2it) \tag{4.4}$$

where $|\mathcal{K}_{\vec\alpha}| = \mathcal{N}_{\vec\alpha\vec\alpha}$.

The Klein bottle amplitude in the *transverse channel* is obtained by performing an S modular transformation

$$\begin{aligned}\mathsf{Z}_K &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}\left(\frac{1}{4\pi^2\alpha' t}\right)^{\frac{D}{2}}\sum_{\vec\alpha}\mathcal{K}_{\vec\alpha}\chi^{\rm susy}_{\vec\alpha}(2it)\\ &= \frac{1}{2}\frac{1}{(8\pi^2\alpha')^{\frac{D}{2}}}\int_0^\infty \frac{dl}{4l}2^D l^{\frac{D}{2}}\sum_{\vec\alpha}\mathcal{K}_{\vec\alpha}\chi^{\rm susy}_{\vec\alpha}\left(\frac{i}{l}\right)\\ &= \frac{1}{2}\frac{1}{(8\pi^2\alpha')^{\frac{D}{2}}}\int_0^\infty \frac{dl}{4}\tilde{\mathcal{Z}}_K(il)\end{aligned} \tag{4.5}$$

where $l = 1/2t$.

$$\tilde{\mathcal{Z}}_K(il) = \frac{1}{2}\sum_{\vec{\alpha}} O^2_{\vec{\alpha}}\chi^{\rm susy}_{\vec{\alpha}}(il) \tag{4.6}$$

represents a closed string propagating between two crosscaps (orientifold planes) and

$$O^2_{\vec{\alpha}} = 2^D \mathcal{K}^{\vec{\beta}} S_{\vec{\beta}\vec{\alpha}}\,. \tag{4.7}$$

As mentioned, generically, massless RR fields will be present such that $\tilde{\mathcal{Z}}_K(il) \propto O^2_{\vec{\alpha}} q^0 + \cdots$ will lead to undesired tadpole divergencies when integrated over l. Therefore one has to include D-brane amplitudes and the corresponding open string sectors to cancel such divergences.

4.2 The open sector: Cylinder and Möbius strip amplitudes

Let us discuss the open sector of the theory. Here the partition function takes the form

$$\mathsf{Z}_{open} = \int_0^\infty \frac{dt}{4t} \mathrm{Tr}_{\mathcal{H}_o}\left[\left(\frac{1+\Omega}{2}\right) e^{2\pi i (L_0 - \frac{c}{24})it}\right] = \mathsf{Z}_C + \mathsf{Z}_M \tag{4.8}$$

where

$$\begin{aligned}\mathsf{Z}_C &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}\mathrm{Tr}_{\mathcal{H}_o}\left[e^{2\pi i(L_0-\frac{c}{24})it}\right] \\ &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}\left(\frac{1}{8\pi^2\alpha' t}\right)^{\frac{D}{2}} \mathrm{Tr}'_{\mathcal{H}_o}\left[e^{2\pi i(L_0-\frac{c}{24})it}\right]\end{aligned}$$

and

$$\begin{aligned}\mathsf{Z}_M &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}\mathrm{Tr}_{\mathcal{H}_o}\left[\Omega e^{2\pi i(L_0-\frac{c}{24})it}\right] \\ &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}\left(\frac{1}{8\pi^2\alpha' t}\right)^{\frac{D}{2}} \mathrm{Tr}'_{\mathcal{H}_o}\left\{\exp\left[-2\pi t(L_0 - \frac{c}{24})\right]\Omega\right\}.\end{aligned} \tag{4.9}$$

Similarly as above Tr' denotes the trace over the discrete oscillator modes and the factor $(8\pi^2\alpha' t)^{-D/2}$ comes from the integral over the bosonic zero modes. The traces can be written in terms of the supersymmetric characters $\chi^{\rm susy}_{\vec{\alpha}}$ as

$$\begin{aligned}\mathcal{Z}_C(it) &= \frac{1}{2}\mathrm{Tr}'_{\mathcal{H}_o}\,[e^{2\pi i(L_0-\frac{c}{24})it}\,] = \frac{1}{2}\sum_{\vec{\alpha}}\mathcal{C}_{\vec{\alpha}}\chi^{\rm susy}_{\vec{\alpha}}(it) \\ \mathcal{Z}_M(it) &= \frac{1}{2}\mathrm{Tr}'_{\mathcal{H}_o}\,[e^{2\pi i(L_0-\frac{c}{24})it}\Omega\,] = \frac{1}{2}\sum_{\vec{\alpha}}\mathcal{M}_{\vec{\alpha}}\hat{\chi}^{\rm susy}_{\vec{\alpha}}(it+\frac{1}{2})\end{aligned} \tag{4.10}$$

where

$$\mathcal{C}^{\vec{\alpha}} = C^{\vec{\alpha}}_{\vec{\alpha}',\vec{\alpha}''} n^{\vec{\alpha}'} n^{\vec{\alpha}''}\,; \quad \mathcal{M}_{\vec{\alpha}} = \sum_{\vec{\alpha}'} M_{\vec{\alpha}\vec{\alpha}'} n^{\vec{\alpha}'} \tag{4.11}$$

represent the multiplicity of states contained in the characters and $n^{\vec{\alpha}}$ are Chan-Paton multiplicities (see discussion below (2.8)). $C^{\vec{\alpha}}_{\vec{\alpha}'\vec{\alpha}''}$ must thus be positive integer numbers whereas $M_{\vec{\alpha}\vec{\alpha}'}$ are integer numbers. *Hatted* characters from the Möbius strip are defined as

$$\hat{\chi}^{\rm susy}_{\vec{\alpha}}\left(it+\frac{1}{2}\right) = e^{-i\pi(\Delta_{\vec{\alpha}}-\frac{Q_{\vec{\alpha}}}{2})}\chi^{\rm susy}_{\vec{\alpha}}\left(it+\frac{1}{2}\right) \tag{4.12}$$

where a phase has been extracted to make them real (see appendix A).

We can therefore proceed to write down such amplitudes in the transverse channel in order to study factorization and tadpole cancellation. A rescaling of the parameter t in each amplitude is needed in order to express such amplitudes in terms of the common tube length $l = -1/2\pi \log q$. While supersymmetric characters in the amplitude from the cylinder involve only S transformations relating the open and closed channels (see (2.10)), the characters in the Möbius strip are evaluated at $it + 1/2$, and thus expressing them in terms of the parameter l requires a combined action of both T and S [26, 27]. In fact, for our characters, it is shown in appendix A that such transformation is achieved by a matrix

$$\hat{P} = T^{(-1/2)} S T^2 S^{-1} T^{(1/2)} \tag{4.13}$$

where

$$T^{(1/2)}_{\vec{\alpha}\vec{\alpha}} = e^{\pi i (\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})} \tag{4.14}$$

is the phase introduced in (4.12) such that characters in the direct and transverse channels are related as

$$\hat{\chi}^{NS/R}_{\vec{\alpha}}\left(it + \frac{1}{2}\right) = (2it)^d \hat{P}_{\vec{\alpha}\vec{\alpha}'} \hat{\chi}^{NS/R}_{\vec{\alpha}'}\left(\frac{i}{4t} + \frac{1}{2}\right). \tag{4.15}$$

Thus

$$\begin{aligned}
\mathsf{Z}_C &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}\left(\frac{1}{8\pi^2\alpha' t}\right)^{\frac{D}{2}} \sum_{\vec{\alpha}} \mathcal{C}_{\vec{\alpha}} \chi^{\rm susy}_{\vec{\alpha}}(it) \\
&= \frac{1}{2}\frac{1}{(8\pi^2\alpha')^{\frac{D}{2}}} \int_0^\infty \frac{dl}{4} \sum_{\vec{\alpha}} \mathcal{C}_{\vec{\alpha}} S_{\vec{\alpha}\vec{\alpha}'} \chi^{\rm susy}_{\vec{\alpha}'}(il) \\
\mathsf{Z}_M &= \frac{1}{2}\int_0^\infty \frac{dt}{4t}\left(\frac{1}{8\pi^2\alpha' t}\right)^{\frac{D}{2}} \sum_{\vec{\alpha}} \mathcal{M}^{\vec{\alpha}} \hat{\chi}^{\rm susy}_{\vec{\alpha}}\left(it + \frac{1}{2}\right) \\
&= \frac{1}{2}\frac{1}{(8\pi^2\alpha')^{\frac{D}{2}}} \int_0^\infty \frac{dl}{4} 2 \times 2^{\frac{D}{2}} \sum_{\vec{\alpha}\vec{\alpha}'} \mathcal{M}^{\vec{\alpha}} i^d \hat{P}_{\vec{\alpha}\vec{\alpha}'} \hat{\chi}^{\rm susy}_{\vec{\alpha}'}\left(il + \frac{1}{2}\right).
\end{aligned} \tag{4.16}$$

where $d = (D-2)/2$.

The sum of all three amplitudes in the transverse channel thus reads

$$\begin{aligned}
\mathsf{Z}_K + \mathsf{Z}_C + \mathsf{Z}_M &= -\frac{1}{2}\frac{1}{(8\pi^2\alpha')^{\frac{D}{2}}} \int_0^\infty \frac{dl}{4} \sum_{\vec{\alpha}} \times \\
&\quad \times \left\{ O^2_{\vec{\alpha}} \chi_{\vec{\alpha}}(il) + D^2_{\vec{\alpha}} \chi_{\vec{\alpha}}(il) + 2 \times 2^{\frac{D}{2}} \tilde{\mathcal{M}}_{\vec{\alpha}} \hat{\chi}_{\vec{\alpha}}\left(il + \frac{1}{2}\right)\right\}.
\end{aligned} \tag{4.17}$$

Therefore the requirement to reconstruct a perfect square for $l \to \infty$, the factorization property sketched in equation (2.6), amounts to

$$D^2_{\vec{\alpha}} + 2 \times 2^{\frac{D}{2}} \tilde{\mathcal{M}}_{\vec{\alpha}} + O^2_{\vec{\alpha}} = \text{perfect square} \tag{4.18}$$

Namely, $2^{\frac{D}{2}} \tilde{\mathcal{M}}_{\vec{\alpha}} = \pm D_{\vec{\alpha}} O_{\vec{\alpha}}$, recalling that $D^2_{\vec{\alpha}}$ is a quadratic polynomial in $n_{\vec{\alpha}}$ whereas $\tilde{\mathcal{M}}_{\vec{\alpha}}$ is a linear polynomial in $n_{\vec{\alpha}}$.

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Moreover, for transverse characters containing RR fields, *i.e.* those originated from the periodic blocks in the direct channel of the Klein bottle and cylinder and the R sector in the direct channel of the Möbius strip, zero RR charge condition (2.7) must be satisfied.

5. Examples in 8 dimensions

We illustrate our construction through explicit examples in $D = 8$ dimensions. In this case $c_{\text{int}} = 3$ and there are only three Gepner models: 1^3, 1 4 and 2^2. Such models are known to correspond to toroidal compactifications on, specific, rational tori [4]. Due to their simplicity, since they involve few blocks and low k (thus a manageable number of states), it proves useful to study them in detail and look for an exhaustive set of solutions. In this section we concentrate in the open sector. Open descendants from toroidal compactification have been discussed in [29] (see also [30, 31] for other perspectives). Other examples in 6 and 4 spacetime dimensions are then presented in forthcoming sections.

Gepner model $\mathbf{1^3}$. For the $k = 1$ minimal model, the labels (l, q, s) in the standard range of the NS sector are (0,0,0); (0,0,2); (1,1,0); (1,1,2); $(1, -1, 0)$ and $(1, -1, 2)$. The corresponding conformal dimensions and charges in all the inequivalent twisted sectors are contained in table 1.

There are three Gepner models containing only products of $k = 1$ minimal models, namely 1^r with $r = 3, 6$ and 9, which define string theories in 8, 6 and 4 dimensions respectively. $\chi^{\text{susy}}_{\vec{\alpha}}$ contains N_1, N_2 and N_3 factors of $\chi_{(0,0)}$, $\chi_{(1,-1)}$ and $\chi_{(1,1)}$ respectively, such that $N_1 + N_2 + N_3 = R$ and $Q_{\vec{\alpha}} = \frac{N_2 - N_3}{3} \in \mathbb{Z}$. Any cyclic permutation of (N_1, N_2, N_3) leads to the same $\chi^{\text{susy}}_{\vec{\alpha}}$.

According to (3.39) the conformal dimensions of the states contained in a given $\chi^{\text{susy}}_{\vec{\alpha}}$ are

$$\Delta_{\vec{\beta}(n)} - \frac{1}{2} = \frac{N_3}{3} \mod 1\,. \tag{5.1}$$

The general form of the states is

$$\left[\prod_{i=1}^{d} (0)^{1-d_i} (2)^{d_i}\right] (0,0,0)^{N_1 - n_1} (0,0,2)^{n_1} (1,-1,0)^{N_2 - n_2} (1,-1,2)^{n_2} (1,1,0)^{N_3 - n_3} (1,1,2)^{n_3}$$

n	Representations	Δ	Q	n	Representations	Δ	Q
0	$(0,0,0)$	0	0	0	$(0,0,2) \sim (1,\pm 3,\pm 4)$	$\frac{3}{2}$	± 1
1	$(0,1,1)$	$\frac{1}{24}$	$\frac{1}{6}$	1	$(0,1,3) \sim (1,-2,-3)$	$\frac{25}{24}$	$-\frac{5}{6}$
2	$(0,2,2) \sim (1,-1,0)$	$\frac{1}{6}$	$\frac{1}{3}$	2	$(0,2,4) \sim (1,-1,-2)$	$\frac{2}{3}$	$-\frac{2}{3}$
3	$(0,3,3) \sim (1,0,1)$	$\frac{3}{8}$	$\frac{1}{2}$	3	$(0,3,5) \sim (1,0,-1)$	$\frac{3}{8}$	$-\frac{1}{2}$
4	$(0,4,4) \sim (1,1,2)$	$\frac{2}{3}$	$\frac{2}{3}$	4	$(0,4,6) \sim (1,1,0)$	$\frac{1}{6}$	$-\frac{1}{3}$
5	$(0,5,5) \sim (1,2,3)$	$\frac{25}{24}$	$\frac{5}{6}$	5	$(0,5,7) \sim (0,-1,-1)$	$\frac{1}{24}$	$-\frac{1}{6}$

Table 1: Conformal weights and charges for the $k = 1$ minimal model.

where the first two entries refer to the label λ of the spacetime contribution, $d_i = 0, 1$ and $n_i = 0, \ldots, N_i$. The odd U(1) charge condition leads to

$$\sum_{i=1}^{d} d_i + n_1 - n_2 + n_3 + \frac{N_2 - N_3}{3} = \text{odd}\,. \tag{5.2}$$

The relation (3.40) in the case 1^3 implies that two characters are identical if the following replacements are performed in each internal theory $(0,0) \to (1,-1); (1,-1) \to (1,1); (1,1) \to (0,0)$. Therefore there are three supersymmetric characters for this model, namely $\chi_A = \chi^{\text{susy}}_{(0,0)^3}$, $\chi_B = \chi^{\text{susy}}_{(0,0)(1,-1)(1,1)}$ and $\chi_C = \chi^{\text{susy}}_{(0,0)(1,1)(1,-1)}$. Where, for instance, (see (3.35))

$$\chi_A = \chi^{\text{susy}}_{(0,0)^3} = \sum_{n,p \bmod 6} \frac{(-1)^{n+p}}{6} e^{2\pi i n^2 \frac{c}{24}\tau} \left[\chi_0\left(\tau, \frac{n}{2}\tau + \frac{p}{2}\right)\right]^3 \chi^3_{(0,0)}\left(\tau, \frac{n}{2}\tau + \frac{p}{2}\right). \tag{5.3}$$

Notice that $\chi^3_{(0,0)}$ is a short hand notation indicating that the same character $\chi_{(0,0)}$ is being considered in each internal block. The conformal dimensions of the highest weight states are $\Delta_A = 1/2$ and $\Delta_B = \Delta_C = 5/6$, respectively and the GSO projected combinations of states contained in these characters are listed in appendix B. Notice, for instance, that $\chi^{\text{susy}}_{(0,0)^3}$ massless states span the $D = 8$, $N = 1$ vector representation[4] [33].

The matrices $S^{(1^3)}$ and $\hat{P}^{(1^3)}$ are

$$S^{(1^3)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \\ 1 & e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \end{pmatrix}; \qquad \hat{P}^{(1^3)} = \frac{i^{-d}}{\sqrt{3}} \begin{pmatrix} -1 & 1 & 1 \\ 1 & e^{-\frac{\pi i}{3}} & e^{\frac{\pi i}{3}} \\ 1 & e^{\frac{\pi i}{3}} & e^{-\frac{\pi i}{3}} \end{pmatrix} \tag{5.4}$$

with $d = (D-2)/2 = 3$

There are two modular invariant combinations of characters in the torus to be considered, diagonal and charge conjugation. We discuss them separately.

i) Diagonal $(\mathbf{1_A})^\mathbf{3}$. There are several possibilities for the Klein bottle partition function in the direct channel, namely

$$\mathcal{Z}_K(it) = \frac{1}{2}\left[\pm\chi_A(2it) \pm \chi_B(2it) \pm \chi_C(2it)\right]. \tag{5.5}$$

Let us start with all positive signs. The partition function in the transverse channel reads

$$\tilde{\mathcal{Z}}_K(il) = \frac{1}{2} 2^8 \sqrt[2]{3} \chi_A(il)\,, \tag{5.6}$$

so only the term

$$\tilde{\mathcal{Z}}_M(il) = \frac{1}{2} \times 2 \times 2^4 \sqrt[2]{3} A \hat{\chi}_A\left(il + \frac{1}{2}\right) \tag{5.7}$$

must be present in the Möbius strip sector. The transverse cylinder amplitude must read

$$\tilde{\mathcal{Z}}_C(il) = \frac{1}{2}\sqrt[2]{3}\left[A^2\chi_A(il) + B^2\chi_B(il) + C^2\chi_C(il)\right] \tag{5.8}$$

to ensure factorization. Since massless RR tadpoles (and also NSNS here) are all contained in χ_A, $A = -16$ is needed in order to achieve tadpole cancellation.

[4] Namely, it contains $1 + 1 + 6 + 4 + 4'$ SO(6) little group representations

When rewriting the amplitudes in the direct channel (using the transformation matrices (5.4)) the linear combinations of coefficients multiplying the three characters must satisfy the following consistency conditions:

a) they must be integer polynomials in n_i;

b) the coefficients of χ_A, the character containing the massless vector in the open sector, must be $\frac{1}{2}n_i(n_i+1)$ for $\mathrm{Sp}(n_i)$, $\frac{1}{2}n_i(n_i-1)$ for $\mathrm{SO}(n_i)$ and n_i^2 for $\mathrm{U}(n_i)$ (or rather $n_i\bar{n}_i$, see footnote 3 in page 7). Consistency conditions *a*) and *b*) imply in this case $A=-n$ and $B=C=0$ leading to the direct channel amplitudes

$$\mathcal{Z}_C(it) = \frac{1}{2}n^2[\chi_A(it)+\chi_B(it)+\chi_C(it)] \tag{5.9}$$

$$\mathcal{Z}_M(it) = \frac{1}{2}n\left[\hat{\chi}_A\left(it+\frac{1}{2}\right)-\hat{\chi}_B\left(it+\frac{1}{2}\right)-\hat{\chi}_C\left(it+\frac{1}{2}\right)\right]. \tag{5.10}$$

Then the tadpole cancellation condition is $n=2^4$, and we are lead to an Sp(16) gauge group with massive matter transforming in the antisymmetric[5] and symmetric representations (recall that the change in sign in MS for different levels changes the symmetry of the corresponding representation).

It is easy to check that other possible combinations of signs in the partition function from the Klein bottle (5.5) do not admit solutions satisfying conditions *a*) and *b*).

ii) Charge conjugation $(\mathbf{1_C})^\mathbf{3}$. The Klein bottle partition function in the direct channel is in this case

$$\mathcal{Z}_K(it) = \frac{1}{2}\chi_A(2it) \tag{5.11}$$

which, written in the transverse channel, reads

$$\tilde{\mathcal{Z}}_K(il) = \frac{1}{2}2^8\frac{1}{\sqrt[2]{3}}(\chi_A(il)+\chi_B(il)+\chi_C(il))\,. \tag{5.12}$$

Generic expressions for cylinder and Möbius strip amplitudes in the transverse channel are given by

$$\tilde{\mathcal{Z}}_M(il) = \frac{1}{2}\times 2\times 2^4\frac{1}{\sqrt[2]{3}}\left[A\hat{\chi}_A\left(il+\frac{1}{2}\right)+B\hat{\chi}_B\left(il+\frac{1}{2}\right)+C\hat{\chi}_C\left(il+\frac{1}{2}\right)\right]$$
$$\tilde{\mathcal{Z}}_C(il) = \frac{1}{2}\frac{1}{\sqrt[2]{3}}[A^2\chi_A(il)+B^2\chi_B(il)+C^2\chi_C(il)]$$

where, A,B,C are linear (complex) combinations of n_i ($i=A,B,C$). Tadpole cancellation thus requires $A=-16$. When rewriting them in the open string direct channel we obtain

$$3\mathcal{Z}_M(it) = \frac{1}{2}\left[(-A+B+C)\hat{\chi}_A\left(it+\frac{1}{2}\right)+(A+Be^{\frac{\pi i}{3}}+Ce^{\frac{-\pi i}{3}})\chi_B\left(it+\frac{1}{2}\right)+\right.$$
$$\left.+(A+Be^{\frac{-\pi i}{3}}+Ce^{\frac{\pi i}{3}})\chi_C\left(it+\frac{1}{2}\right)\right]$$

[5] Actually in $\mathbf{119}+\mathbf{1}$ since the antisymmetric representation is reducible

$$
\begin{aligned}
3\mathcal{Z}_C(it) = \frac{1}{2}\Big[&(A^2+B^2+C^2)\chi_A(it)+(A^2+B^2e^{-\frac{2\pi i}{3}}+C^2e^{\frac{2\pi i}{3}})\chi_B(it)+\\
&+(A^2+B^2e^{\frac{2\pi i}{3}}+C^2e^{-\frac{2\pi i}{3}})\chi_C(it)\Big]\,.
\end{aligned}\tag{5.13}
$$

A solution satisfying the consistency conditions $a)$ and $b)$ is $B=-ne^{\frac{i}{3}\pi}-\bar{n}e^{-\frac{i}{3}\pi}+m=C^*$, $A=-n-\bar{n}-m$, (numerically $n=\bar{n}$) leading to

$$
\begin{aligned}
\mathcal{Z}_C(it) &= \left(\frac{1}{2}m^2+n\bar{n}\right)\chi_A(it)+\left(\frac{1}{2}n^2+nm\right)\chi_B(it)+\left(\frac{1}{2}\bar{n}^2+\bar{n}m\right)\chi_C(it)\\
\mathcal{Z}_M(it) &= \frac{1}{2}m\hat{\chi}_A\left(it+\frac{1}{2}\right)-\frac{1}{2}\bar{n}\hat{\chi}_B\left(it+\frac{1}{2}\right)-\frac{1}{2}n\hat{\chi}_C\left(it+\frac{1}{2}\right)
\end{aligned}\tag{5.14}
$$

with the tadpole cancellation condition

$$
-A=n+\bar{n}+m=2n+m=16\,. \tag{5.15}
$$

The interpretation here is less clear. n and $\bar{n}$ have been interchanged in $\mathcal{Z}_M$ with respect to what appears in $\mathcal{Z}_C$. On the other hand, the characters χ_B and χ_C are the same (considered as functions of τ and z) so, expansion in powers of q seems to indicate an Sp$(2(8-n))\times$ U(n) $N=1$, $D=8$ vector multiplet with descendant massive fields in the corresponding adjoint representations and extra massive matter transforming in $(\mathbf{2(8-n)},\mathbf{n})+(\mathbf{2(8-n)},\bar{\mathbf{n}})+(\mathbf{1},\genfrac{}{}{0pt}{}{\square}{\square})+(\mathbf{1},\overline{\genfrac{}{}{0pt}{}{\square}{\square}})$ (or $(1,\square\square)+(1,\overline{\square\square})$ depending on the mass level).

Such exchange of n and $\bar{n}$ might indicate that linear combinations (symmetric and antisymmetric) of fields $|\mathrm{B}>$ and $|\mathrm{C}>$, which have similar conformal weight and charge, must be considered.

Notice that $n=0$ leads to Sp(16) and there is no contribution from massive characters χ_B and χ_C.

A particular prescription leading to a consistent theory, *i.e.* a theory verifying the requirements of factorization and tadpole cancellation, was found in [26] for the charge conjugation torus partition function. In this case, Cardy [34] has shown that, when there are the same number of characters than Chan Paton factors, a natural solution for the partition function from the cylinder is given by $C^c_{ab}=N^c_{ab}$, where N^c_{ab}, the number of conformal blocks of a rational CFT, can be written in terms of the S matrix as

$$
N^c_{ab}=\sum_l \frac{S_{al}S_{bl}(S^\dagger)^{lc}}{S_{0l}}\,. \tag{5.16}
$$

This is the celebrated Verlinde theorem [35] arising as a consequence of the well established fact that the modular matrix S diagonalizes the fusion rules. The proof of the theorem relies on the technical assumption that both left and right extended chiral algebras consist only of generators with integral conformal dimension and thus evidently excludes the superconformal case. A generalized Verlinde formula which describes the fusion rules in all sectors of $N=1$ superconformal theories was obtained in [36] whereas the $N=2$ case was dealt with in reference [37]. Cardy's extension of the theorem to surfaces with boundaries was generalized to $N=1$ in reference [38].

It was shown in [20] that one can define another integer valued object [39]

$$Y_{ab}^{c} = \sum_{l} \frac{S_{al} P_{bl} (P^{\dagger})^{lc}}{S_{0l}} \tag{5.17}$$

leading to

$$\mathcal{K}_a = Y_{a0}^{0} \,, \qquad M_a^b = Y_{a0}^{b} \tag{5.18}$$

This ansatz requires symmetric S and P matrices, and therefore non trivial solutions must reproduce some of the models found above. Actually, the partition functions (5.14) can be obtained with this prescription.

There are other solutions satisfying the factorization and tadpole cancellation conditions but they do not verify the requirement $b)$ above, *i.e.* they do not lead to massless vectors transforming in either Sp, SO or U groups.

Gepner model $\mathbf{2^2}$. All the representations of the $N = 2$ superconformal $k = 2$ minimal model can be obtained by twisting the pairs $(0,0,0);(0,0,2)$ and $(1,-1,0);(1,-1,2)$. They are listed in appendix B. The characters of the Virasoro algebra are contained in table 2 and the independent supersymmetric characters in the 2^2 product theory are $\chi_A = \chi^{\text{susy}}_{(0,0)^2}$, $\chi_B = \chi^{\text{susy}}_{(1,-1);(1,1)}$, $\chi_C = \chi^{\text{susy}}_{(0,0);(2,0)}$ with $\Delta_A = 1/2$, $\Delta_B = 3/4$, $\Delta_C = 1$, respectively. The GSO projected combinations of states contained in these characters are listed in appendix B.

Character	Weight	Charge
$(0,0)$	0	0
$(2,-2)$	$\frac{1}{4}$	$\frac{1}{2}$
$(2,0)$	$\frac{1}{2}$	0
$(2,2)$	$\frac{1}{4}$	$-\frac{1}{2}$
$(1,-1)$	$\frac{1}{8}$	$\frac{1}{4}$
$(1,1)$	$\frac{1}{8}$	$-\frac{1}{4}$

Table 2: Conformal weights and charges for the $k = 2$ minimal model.

This model admits only the diagonal modular invariant. The Klein bottle partition function is either

$$\begin{aligned}
&1) \quad \mathcal{Z}_K(it) = \frac{1}{2}\left[\chi_A(2it) + \chi_B(2it) + \chi_C(2it)\right],\\
&2) \quad \mathcal{Z}_K(it) = \frac{1}{2}\left[-\chi_A(2it) + \chi_B(2it) + \chi_C(2it)\right] \quad \text{or}\\
&3) \quad \mathcal{Z}_K(it) = \frac{1}{2}\left[\chi_A(2it) + \chi_B(2it) - \chi_C(2it)\right].
\end{aligned} \tag{5.19}$$

The matrices S and $\hat{P}$ are in this case

$$S^{(2^2)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}; \qquad \hat{P}^{(2^2)} = i^{-d}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.20}$$

with $d = (D-2)/2 = 3$.

They may be used to rewrite the Klein bottle amplitude in the transverse channel as

$$1) \quad \tilde{\mathcal{Z}}_K(il) = 2^8 \chi_A(il) \tag{5.21}$$

$$2) \quad \tilde{\mathcal{Z}}_K(il) = \frac{1}{2} 2^8 \left[\chi_A(il) - \chi_B(il) - \chi_C(il)\right] \tag{5.22}$$

$$3) \quad \tilde{\mathcal{Z}}_K(il) = 2^8 \chi_B(il)\,. \tag{5.23}$$

Let us discuss case 1) first. The contribution from the open sector must then be

$$\tilde{\mathcal{Z}}_M\left(il+\frac{1}{2}\right) = -2\times 2^4 A\,\hat{\chi}_A\left(il+\frac{1}{2}\right) \tag{5.24}$$

$$\tilde{\mathcal{Z}}_C(il) = A^2\,\chi_A(il)+B^2\chi_B(il)+C^2\chi_C(il)\,. \tag{5.25}$$

Consistency conditions lead to $A=\frac{1}{2}(n+\bar{n})=B$, $C=\frac{i}{2}(n-\bar{n})$ (recall $n=\bar{n}$) and $n=16$ from tadpole cancellation. Therefore

$$\mathcal{Z}_C(it) = n\bar{n}\chi_A(it)+\left(\frac{1}{2}n^2+\frac{1}{2}\bar{n}^2\right)\chi_B(it) \tag{5.26}$$

and

$$\mathcal{Z}_M\left(it+\frac{1}{2}\right) = -\frac{1}{2}(n+\bar{n})\hat{\chi}_B\left(it+\frac{1}{2}\right). \tag{5.27}$$

Here we obtain a massless $N=1, D=8$ U(16) vector multiplet (with massive descendants in the adjoint) and massive states in the antisymmetric representation (with descendants in the symmetric or antisymmetric according to the level).

Let us now consider case 2). Since all characters contribute to the Klein bottle amplitude, it appears that complex multiplicities are not permitted and therefore no unitary groups are allowed. The resultant partition functions in the transverse channel are

$$\tilde{\mathcal{Z}}_M\left(il+\frac{1}{2}\right) = -2\times 2^4 n\,\hat{\chi}_A\left(il+\frac{1}{2}\right) \tag{5.28}$$

$$\tilde{\mathcal{Z}}_C(il) = 2n^2\,\chi_A(il) \tag{5.29}$$

whereas in the direct channel they are

$$\mathcal{Z}_M(it+\frac{1}{2}) = -n\hat{\chi}_B(it+\frac{1}{2}) \tag{5.30}$$

$$\mathcal{Z}_C(it) = n^2[\chi_A(it)+\chi_B(it)+\chi_C(it)]. \tag{5.31}$$

The tadpole cancellation condition is now $n=8$. Even though the massless character (i.e. the one corresponding to the massless state) does not appear in the Möbius strip amplitude, there is no consistent solution in terms of n and $\bar{n}$ multiplicities and, therefore, no unitary group seems to be allowed. The amplitude can be interpreted as corresponding to SO(8) $\times$ Sp(8) vector multiplet plus massive descendant states in symmetric and antisymmetric representations and in bi-fundamentals.

Starting with $\mathcal{Z}_K=\chi_A-\chi_B+\chi_C$ the same result is obtained.

The third case is interesting since the Klein bottle amplitude has no massless tadpoles. Therefore, closed unoriented theory is consistent with no need of an open string sector. (See [11, 41, 42] for other examples).

Gepner model 4 1. The states of the $k=4$ minimal model may be classified in two sets, one with $l=0$ and $l=4$ and the other one with $l=2=\frac{k}{2}$ (short). They are listed in appendix B.

Taking into account the spectrum of the $k = 1$ minimal model, the only possible combinations of states in the 4 1 Gepner model are

$$\begin{array}{|l|l|} \hline \chi^{\text{susy}}_{\vec{\alpha}} & \Delta_{\vec{\beta}(n)} \bmod 1 \\ \hline \chi^{\text{susy}}_{(0,0)(0,0)} & \frac{1}{2} \\ \hline \chi^{\text{susy}}_{(2,0)(0,0)} & \frac{5}{6} \\ \hline \end{array} \tag{5.32}$$

Let us introduce the following notation: $\chi_A \equiv \chi^{\text{susy}}_{(0,0)(0,0)}, \chi_B \equiv \chi^{\text{susy}}_{(2,0)(0,0)}$ where the first pair of indices refer to $k = 4$ and the second one to $k = 1$.

In fact, it may be seen by comparing the tables in appendix B that the spectrum of states in χ_A is identical to that of $\chi_{(0,0)^3}$ in the 1^3 model and similarly for χ_B and $\chi_{(0,0)(1,-1)(1,1)}$ (or equivalently $\chi_{(0,0)(1,1)(1,-1)}$). Actually, this is in agreement with the conjectured equivalence between conformal models $\mathbf{4} \equiv \mathbf{1^2_A}$ [43].

Moreover, this identification remains valid for the open string. This can be easily checked by noticing that there is only one modular invariant combination of characters in this case that leads to the following Klein bottle amplitude in the direct channel

$$\mathcal{Z}_K(it) = \frac{1}{2}\left[\chi_A(2it) + \chi_B(2it)\right]. \tag{5.33}$$

Matrices S and $\hat{P}$ in this case are

$$S^{(4\ 1)} = \frac{1}{\sqrt[2]{3}}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}; \qquad \hat{P}^{(4\ 1)} = \frac{i^{-d}}{\sqrt[2]{3}}\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \tag{5.34}$$

where $d = (D-2)/2 = 3$, the first row and column refer to A and the second ones to B. They allow us to find the transverse channel contribution from the Klein bottle

$$\tilde{\mathcal{Z}}_K(il) = \frac{1}{2} 2^8 \sqrt[2]{3}\chi_A(il) \tag{5.35}$$

(compare with (5.7)) and to find that consistency requirements lead to the same Sp(16) theory found in the $(\mathbf{1_A})^{\mathbf{3}}$ model.

Before concluding this section let us stress some aspects of our results.

Notice that, while rank 16 groups can be obtained $-$*e.g.* in the 2^2 model, case 1) $-$, gauge groups of rank 8 are also found. Naively, a rank $N_D = 16$ group could have been expected from a toroidal compactification of SO(32) string with 16 pairs of 9 branes.

This subject has been discussed from several perspectives in the literature. In fact, rank reduction can be explained from the presence of a discrete $NSNS$ antisymmetric field (see [29] and references therein). Indeed, while generically orientifold projection kills antisymmetric field fluctuations, in some cases a discrete vacuum expectation value for a rank b antisymmetric tensor field can still be preserved, leading to $N_D = 32 \times 2^{-\frac{b}{2}}$. The issue of rank reduction was addressed in [30, 44] (see also [31, 32]) from a more topological point of view. There, the presence of four fixed orientifold seven planes of different kind, O^+ (O^-) carrying -8 ($+8$) units of D-brane charge, is identified depending on the way the orientifold acts on the T^2 torus (with or without vector structure). While D-branes on smooth points generate unitary groups, when they sit on the top of a O^- (O^+) orientifold plane an Sp (SO) gauge group is produced.

Thus, the $\mathrm{Sp}(2(8-n))\times \mathrm{U}(n)$ group obtained in the 1^3 example above would correspond to the case of $3O^+ + O^-$ points, where $8-n$ pairs sit at O^- points and the rest coincide at smooth points.

Also, the U(16) group in 2^2 would correspond to branes at smooth points in a $4O^+$ configuration. Recall that sign change in the Klein bottle amplitude for this case leads to rank reduction, here with $\mathrm{Sp}(8)\times\mathrm{SO}(8)$, which would correspond to two groups of four pairs of branes distributed between the O^+ and O^- points. The third case in (5.19), with no tadpoles, would correspond in this description to a $2O^+ + 2O^-$ configuration. Notice that correspondence between our results and this description is indicative and would require further investigation.

6. Examples in 6 dimensions

$N=1$, $D=6$ supersymmetric models have potential chiral and gravitational anomalies. Anomaly cancellation is thus a strong check on the consistency of the whole construction.

The massless representations in $D=6$ are labelled by the representations of the little group $\mathrm{SO}(4)\sim\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ and they gather into gravity, vector, tensor and hyper multiplets. Clearly gravity and tensor multiplets are present only in the closed sector. They are obtained by plugging left and right moving massless states, which are invariant under Ω orientation reversal exchange. Consider, as an example, the coupling of RR states. Massless Ramond states in $D=6$ are conformal weight 1/2 states of the form (see table 12) $|(+,+)_1,R_0>,|(-,-)_{-1},R_0>$ which organize into $(\mathbf{2},\mathbf{1})R$ Lorentz representation or $|(+,-)_0,R_1>,|(-,+)_0,R_{-1}>$ leading to $(\mathbf{1},\ \mathbf{2})$ representations. We indicate with a subindex the corresponding spacetime or internal charge and R summarizes the internal sector fermionic content. In particular, if left and right R,R' Ramond states couplings $(\mathbf{2},\mathbf{1})R\times(\mathbf{2},\mathbf{1})R'$ are allowed by modular invariant coefficients, orientifolding will lead to $(\mathbf{3},\mathbf{1})\frac{1}{2}(R\times R'+R'\times R)$ triplets and $(\mathbf{1},\mathbf{1})\frac{1}{2}(R\times R'-R'\times R)$ singlets. Together with NSR fermions such states group in a tensor supermultiplet $T=(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1})+2(\mathbf{2},\mathbf{1})$. Recall that, due to anticommutation of fermion fields, only the scalar will be present if $R=R'$. The other closed states are constructed following the same steps.

As an illustration of a specific computation, let us consider the simple 1^6 models. Left and right moving sectors can be coupled in several manners. In particular, each left block can be coupled either diagonally or to its charge conjugate left state (we choose positive $\mathcal{K}_{\vec{\alpha}}$ signs here). Thus, $(\mathbf{1_A})^\mathbf{6}$, $(\mathbf{1_A})^\mathbf{5}\mathbf{1_C}$, $\mathbf{1_A}(\mathbf{1_C})^\mathbf{5}$, $(\mathbf{1_A})^\mathbf{3}(\mathbf{1_C})^\mathbf{3}$, $(\mathbf{1_A})^\mathbf{2}(\mathbf{1_C})^\mathbf{4}$, $(\mathbf{1_A})^\mathbf{4}(\mathbf{1_C})^\mathbf{2}$ and $(\mathbf{1_C})^\mathbf{6}$ modular invariant couplings are possible. The first four cases have $0T+21H$, the last *conjugate* invariant has $10T+11H$, while the other two contain $6T+15H$ multiplets in the closed sector. Simple solutions for some of these cases have been found in [11]. For instance, by noticing that in $(\mathbf{1_A})^\mathbf{6}$ the Klein bottle amplitude in the transverse channel contains just the $q=0$ states,[6] namely

$$\tilde{\mathcal{Z}}_K(il)=9\frac{1}{2}2^6\chi^{\mathrm{susy}}_{(0,0)^6}(il)\,,\tag{6.1}$$

[6] Actually, it is a general result for any k model that diagonal invariant coupling leads to zero charge characters in the transverse channel.

it is immediate to see that $\tilde{\mathcal{Z}}_C(il) = n^2 \tilde{\mathcal{Z}}_K(il)$ and $\tilde{\mathcal{Z}}_M(il) = -2 \times 2^3 n \tilde{\mathcal{Z}}_K(il + 1/2)$ give a consistent theory with $n = 8$ corresponding to SO(8) gauge group and 10 hypermultiplets in the **28** representation.

As another illustration, with a more general solution, let us consider the $(\mathbf{1_A})^\mathbf{4}(\mathbf{1_C})^\mathbf{2}$ model with Klein bottle amplitude

$$\mathcal{Z}_K(it) = \frac{1}{2}\left\{\left(\chi_{(0,0)} + \chi_{(1,-1)} + \chi_{(1,1)}\right)^4 \chi^2_{(0,0)}\right\}^{\mathrm{susy}} \tag{6.2}$$

where only internal blocks are displayed and susy means that sums over n, p, as indicated in (3.35), must be performed. The following contributions from the direct channel of the cylinder

$$\begin{aligned}
\mathcal{Z}_C(it) \;=\; & \left(\frac{1}{2}n_1^2 + n_2^2\right)\Big\{\chi^6_{(0,0)} + \underline{\chi_{(1,-1)}\chi_{(1,1)}\chi^2_{(0,0)}}\,\chi^2_{(0,0)} + \underline{\chi^3_{(1,-1)}\chi_{(0,0)}}\,\chi^2_{(0,0)} + \\
& \qquad + \underline{\chi^3_{(1,1)}\chi_{(0,0)}}\,\chi^2_{(0,0)} + \underline{\chi^2_{(1,-1)}\chi^2_{(1,1)}}\,\chi^2_{(0,0)}\Big\}^{\mathrm{susy}} + \\
& + \left(\frac{1}{2}{n_2}^2 + n_1 n_2\right)\Big\{\chi^4_{(0,0)}\chi_{(1,-1)}\chi_{(1,1)} + \underline{\chi^2_{(0,0)}\chi_{(1,-1)}\chi_{(1,1)}}\;\underline{\chi_{(1,-1)}\chi_{(1,1)}} + \\
& \qquad + \underline{\chi^2_{(1,1)}\chi^2_{(0,0)}}\;\underline{\chi_{(1,1)}\chi_{(0,0)}}\Big\}^{\mathrm{susy}}
\end{aligned} \tag{6.3}$$

and from the Möbius strip

$$\mathcal{Z}_M(it) = -\sum_{all\ characters}\left[(-1)^{N_{(1,1)}}\left(\frac{1}{2}n_1\hat{X}_1 + \frac{1}{2}n_2\hat{X}_2\right)\right]^{\mathrm{susy}}, \tag{6.4}$$

where X_1 and X_2 refer to all the products of characters with the last two factors being respectively $\chi^2_{(0,0)}$ and $\chi_{(1,-1)}\chi_{(1,1)}$ or $\chi_{(1,1)}\chi_{(1,-1)}$, lead to a consistent solution provided the tadpole cancellation condition

$$2n_2 + n_1 = 8 \tag{6.5}$$

is satisfied.

From $\chi^{\mathrm{susy}}_{(0,0)^6}$ we read the $\mathrm{SO}(n_1) \times \mathrm{U}(n_2)$ gauge group. The other massless characters in X_1 (namely $\chi^3_{(1,-1)}\chi^3_{(0,0)}$, $\chi^3_{(1,1)}\chi^3_{(0,0)}$ and any permutation of the first four characters in the products) transform in the adjoint representation of $\mathrm{SO}(n_1)$ and $\mathrm{U}(n_2)$. The massless characters in X_2 (namely $\underline{\chi^2_{(1,-1)}\chi^2_{(1,1)}}\chi_{(1,1)}\chi_{(1,-1)}$ and $\underline{\chi^2_{(1,-1)}\chi^2_{(1,1)}}\chi_{(1,-1)}\chi_{(1,1)}$) have massless states transforming in the antisymmetric representation (descendants will be in the symmetric or antisymmetric, according to the level) of $\mathrm{U}(n_2)$ and a bi-fundamental representation.

Thus, the massless multiplets are

$$\begin{aligned}
&\text{Vector } \ \mathrm{SO}(n_1) \times \mathrm{U}(n_2) \\
&\text{Hypers } \ 4(\boxminus, 1) + 4(1, \mathrm{Adj}) + 6(1, \boxminus) + 6(\mathbf{n}_1, \mathbf{n}_2)\,.
\end{aligned} \tag{6.6}$$

It is easy to check that all gauge and gravitational anomalies cancel (recall that 6 tensor multiplets and 15 hypermultiplets are present in the closed sector) whenever the tadpole cancellation condition (6.5) is satisfied.

It is interesting to compare this computation with an orbifold like case. Consider, for instance, a type-IIB orientifold on T^4/Z_3. By looking at Z_3 twists action on D9-branes one finds (when no Wilson lines are turned on) that the general, massless spectrum reads (see for instance [45])

$$\begin{aligned}&\text{Vector } \mathrm{SO}(n_1) \times \mathrm{U}(n_2)\\ &\text{Hyper } (\mathbf{n}1, \mathbf{n}_2) + (1, \begin{smallmatrix}\square\\\square\end{smallmatrix})\,.\end{aligned} \tag{6.7}$$

If *twisted* tadpole cancellation condition $n_2 - n_1 = 8$ is satisfied, the spectrum is free of both gauge and gravitational anomalies (once the eleven hyper and the ten tensor multiplets from the closed sector are included) independently of the total number of branes. It is the global untwisted RR tadpole cancellation condition $n_1 + 2n_2 = 32$, which fixes the total number of branes.[7] In our case we see that factorization, in massless and massive transverse channel sectors, leads to an effective field theory which is inconsistent unless the number of branes is restricted to $n_1 + 2n_2 = 8$. Thus global information appears to be required at every step in our construction.

7. Examples in 4 dimensions

The 1^9 case. The analysis of this model follows very closely the 1^6 case with the obvious modifications (see [11] for a previous discussion of this model). The diagonal modular invariant partition function $(\mathbf{1_A})^9$ leads to the usual tadpole cancellation condition, $n = 2^{D/2} = 4$ and the massless vector belongs to a Sp(4), $D = 4$, N=1 vector multiplet.

The $(\mathbf{1_A})^7(\mathbf{1_C})^2$ case is analogous to the $(\mathbf{1_A})^4(\mathbf{1_C})^2$ example above. There is a consistent model with Chan Paton gauge group $\mathrm{Sp}(n_2) \times \mathrm{U}(n_1)$ and tadpole cancellation condition $2n_1 + n_2 = 4$.

The 3^5 case. The representations of the $k = 3$ minimal model are listed in appendix B. It is convenient to denote the supersymmetric characters in this theory with the number N_i which refers to the multiplicity of the i-th character in the product. The index $i = 1, \ldots, 9$ refers to the states (0,0), (3,-3), (3,-1), (3,1), (3,3), (2,0), (2,2), (1,-1), (1,1), (2,-2) respectively. The combinations of characters with integer U(1) charge are the following:

$$\begin{aligned}
N_1 + N_6 &= 5\\
N_1 + N_6 &= 3; N_2 + N_7 = N_5 + N_{10} = 1\\
N_1 + N_6 &= 3; N_3 + N_8 = N_4 + N_9 = 1\\
N_1 + N_6 &= N_2 + N_7 = N_3 + N_8 = N_4 + N_9 = N_5 + N_{10} = 1\\
N_1 + N_6 &= 1; N_2 + N_7 = N_5 + N_{10} = 2\\
N_1 + N_6 &= 1; N_3 + N_8 = N_4 + N_9 = 2\,.
\end{aligned} \tag{7.1}$$

[7] Interestingly enough such conditions can be understood, by considering D-brane probes, as consistency conditions of the effective theory in all topological sectors [45]

The diagonal modular invariant partition function in the torus leads to the following expression for the direct channel from the Klein bottle

$$\mathcal{Z}_K(it) = \frac{1}{2}\frac{1}{5}\left[(\chi_I(it) + \chi_{II}(it))^5\right]^{\text{susy}} \tag{7.2}$$

where

$$\begin{aligned} \chi_I &= \chi_{(0,0)} + \chi_{(3,-3)} + \chi_{(3,-1)} + \chi_{(3,1)} + \chi_{(3,3)} \\ \chi_{II} &= \chi_{(2,0)} + \chi_{(2,2)} + \chi_{(1,-1)} + \chi_{(1,1)} + \chi_{(2,-2)} \,. \end{aligned} \tag{7.3}$$

Applying the S matrix for the $k = 3$ model (see appendix B) we find in the transverse channel

$$\tilde{\mathcal{Z}}_K(il) = \frac{1}{2} 2^4 \sqrt[4]{5}\left[(\kappa^{\frac{3}{2}}\tilde{\chi}_{(0,0)}(il) + \kappa^{-\frac{3}{2}}\tilde{\chi}_{(2,0)}(il))^5\right]^{\text{susy}} \tag{7.4}$$

where $\kappa \equiv \frac{1}{2}(1+\sqrt{5})$.

Following the same procedure as in the 6 dimensional examples treated above, we propose the following partition function for the transverse channel from the cylinder

$$\tilde{\mathcal{Z}}_C(il) = \frac{1}{2}\sqrt[4]{5}\mathcal{A}_{\vec{\gamma}}\left[\prod_{i=1}^{5}(\tilde{\chi}_{(0,0)}(il))^{1-\gamma_i}(\tilde{\chi}_{(2,0)}(il))^{\gamma_i}\right]^{\text{susy}} \tag{7.5}$$

Here $\vec{\gamma} = \vec{\gamma}(\vec{\alpha})$ denotes a 5 component vector (one for each theory) taking values 0 or 1 if the state belongs to group I or II respectively. Note that $\vec{\gamma}(\vec{\alpha}_{(n)}) = \vec{\gamma}(\vec{\alpha})$ since the states remain in the same group under twisting.

Therefore the tadpole cancellation conditions are

$$\mathcal{A}_{\vec{0}} = 2^4\kappa^{\frac{15}{2}}\,; \qquad \mathcal{A}_{\vec{1}} = 2^4\kappa^{-\frac{15}{2}} \tag{7.6}$$

where $\vec{0} \equiv (0,0,0,0,0)$ and $\vec{1} \equiv (1,1,1,1,1)$.

Applying the S matrix we can transform (7.5) to the direct channel where the partition function from the cylinder reads

$$\mathcal{Z}_C(it) = \frac{1}{2}\sum_{\vec{\gamma},\vec{\alpha}} \mathcal{C}_{\vec{\gamma}(\vec{\alpha})}\chi_{\vec{\alpha}} \tag{7.7}$$

with

$$\mathcal{C}_{\vec{\gamma}} = \frac{1}{(\sqrt[2]{5}\kappa)^{\frac{5}{2}}}\sum_{\vec{\gamma}'}\frac{1}{(\sqrt[4]{5})^5}\kappa^{(\vec{\gamma}-\vec{\gamma}')^2}(-1)^{\vec{\gamma}\cdot\vec{\gamma}'}\mathcal{A}_{\vec{\gamma}'}\,. \tag{7.8}$$

Let us denote

$$M_{\vec{\gamma}\,\vec{\gamma}'} \equiv \frac{1}{(\sqrt[2]{5}\kappa)^{\frac{5}{2}}}\kappa^{(\vec{\gamma}-\vec{\gamma}')^2}(-1)^{\vec{\gamma}\cdot\vec{\gamma}'} \tag{7.9}$$

the 32×32 real matrix relating $\mathcal{C}_{\vec{\gamma}}$ to $\mathcal{A}_{\vec{\gamma}'}$. It verifies $M^{-1} = M$ and $M = M^{\top}$ and thus it can be used to compute the coefficients in the direct channel using Cardy's prescription [34] ($\mathcal{C}_{\vec{\gamma}} \equiv \sum_{\vec{\alpha},\vec{\beta}} N_{\vec{\alpha}\,\vec{\beta}}^{\vec{\gamma}} n^{\vec{\alpha}} n^{\vec{\beta}}$) with

$$N_{\vec{\alpha}\,\vec{\beta}}^{\vec{\gamma}} = \sum_{\vec{\delta}}\frac{M_{\vec{\alpha}\,\vec{\delta}}M_{\vec{\beta}\,\vec{\delta}}M_{\vec{\gamma}\,\vec{\delta}}}{M_{\vec{0}\,\vec{\delta}}} \tag{7.10}$$

and then

$$\frac{1}{\left(\sqrt[4]{5}\right)^5}\mathcal{A}_{\vec{\gamma}} = \frac{\left(\sum_{\vec{\delta}} M_{\vec{\gamma}\vec{\delta}} n^{\vec{\delta}}\right)^2}{M_{\vec{0}\vec{\gamma}}} . \tag{7.11}$$

We find

$$\begin{aligned} N_{\vec{\alpha}\,\vec{\beta}}^{\vec{\gamma}} &= \prod_{i=1}^{5}(1-\delta_{\alpha_i+\beta_i+\gamma_i,1}) \\ \mathcal{A}_{\vec{\gamma}} &= \frac{1}{\kappa^{5/2}}\frac{\left(\sum_{\vec{\delta}} \kappa^{(\vec{\gamma}-\vec{\delta})^2}(-1)^{\vec{\gamma}.\vec{\delta}} n^{\vec{\delta}}\right)^2}{\kappa^{(\vec{\gamma})^2}} \end{aligned} \tag{7.12}$$

and $\mathcal{A}_{\vec{1}-\vec{\gamma}}$ can be obtained from $\mathcal{A}_{\vec{\gamma}}$ inverting the sign of $\sqrt{5}$ in all powers of κ.

The tadpole cancellation conditions become

$$\mathcal{A}_{\vec{0}} = 2^4\kappa^{\frac{15}{2}} = \kappa^{-\frac{5}{2}}\left(\sum_{\vec{\delta}} \kappa^{(\vec{\delta})^2} n_{\vec{\delta}}\right)^2 \tag{7.13}$$

$$\mathcal{A}_{\vec{1}} = 2^4\kappa^{-\frac{15}{2}} = \kappa^{\frac{5}{2}}\left(\sum_{\vec{\delta}} (-\kappa)^{-(\vec{\delta})^2} n_{\vec{\delta}}\right)^2 , \tag{7.14}$$

and they lead to the following equations for the Chan Paton coefficients

$$\begin{aligned} n_0 + n_2 + n_3 + 2n_4 + 3n_5 &= 12 \\ n_1 + n_2 + 2n_3 + 3n_4 + 5n_5 &= 20 \end{aligned} \tag{7.15}$$

where $n_i = \sum_{\vec{\gamma}\ /\ |\vec{\gamma}|=i} n_{\vec{\gamma}}$ (*e.g.*, $n_4 = n_{(1,1,1,1,0)} + n_{(1,1,1,0,1)} + n_{(1,1,0,1,1)} + n_{(1,0,1,1,1)} + n_{(0,1,1,1,1)}$).

The coefficients (of the real characters $\hat{\chi}$) $\tilde{\mathcal{M}}_{\vec{\gamma}}$ which complete a perfect square in the transverse channel from the Möbius strip are given by

$$\tilde{\mathcal{M}}_{\vec{\gamma}} = \pm\sqrt[4]{5}\sqrt{\mathcal{A}_{\vec{\gamma}}\kappa^{\frac{15}{2}-3(\vec{\gamma})^2}} \tag{7.16}$$

and one of the solutions to the tadpole cancellation conditions is

$$\tilde{\mathcal{M}}_{\vec{\gamma}} = -\sum_{\vec{\delta}} \sqrt[4]{5}(-1)^{(\vec{\gamma})^2}\kappa^{(\vec{\gamma}-\vec{\delta})^2}(-1)^{\vec{\gamma}.\vec{\delta}}\kappa^{\frac{5}{2}-2(\vec{\gamma})^2} n_{\vec{\delta}} . \tag{7.17}$$

Transforming to the direct channel with $\hat{P}^{(k=3)}$ we find

$$M_{\vec{\beta}}^{\vec{\delta}} = -\frac{(-1)^{N_1+N_2+1}}{5}\prod_{i=1}^{5}(1-\delta_{2\delta_i+\gamma_i(\beta_i),1}) . \tag{7.18}$$

The factor $1/5$ cancels when summing over all twisted sectors and therefore the partition function from the Möbius strip in the direct channel is

$$\mathcal{Z}_M = -\frac{1}{2}{\sum_{\vec{\alpha}}}' \left(\prod_{i=1}^{5}(1-\delta_{2\delta_i+\gamma_i(\alpha_i),1})\right)(-1)^{N_1+N_2+1}\hat{\chi}_{\vec{\alpha}}^{\text{susy}}\ n_{\vec{\delta}} . \tag{7.19}$$

Comparing with the contribution from the cylinder

$$\mathcal{Z}_C = \frac{1}{2}\sum_{\vec\beta}{}' \left(\prod_{i=1}^{5}(1-\delta_{\gamma_i(\alpha_i)+\beta_i+\delta_i,1})\right)\chi^{\rm susy}_{\vec\alpha}\; n_{\vec\beta} n_{\vec\delta} \tag{7.20}$$

it is easy to see that the term $n_{\vec\delta}$ appears in (7.19) only if the term $(n_{\vec\delta})^2$ appears in (7.20) and thus the Chan-Paton contribution from $n_{\vec\delta}$ is either vanishing or $\frac{1}{2}n_{\vec\delta}(n_{\vec\delta}\pm 1)$. The sign ± 1 depends on the phase $(-1)^{N_1+N_2+1}e^{-\pi i(\Delta_{\vec\beta}-\frac{Q_{\vec\beta}}{2})}e^{\pi i(\Delta^{GSO}-\frac{1}{2})}$. It is -1 for the massless vector character $\chi_{(0,0)^5}$ and therefore the Chan Paton group is $\mathrm{SO}(n_i)$ for all n_i. It is also -1 for the characters $\chi_{(0,0)^3(3,-3)(2,-2)}$, $\chi_{(2,0)^5}$, $\chi_{(0,0)(1,-1)^3(2,-2)}$, $\chi_{(0,0)^2(3,3)(1,1)^2}$ and it is $+1$ for $\chi_{(0,0)^3(3,3)(2,2)}$, $\chi_{(0,0)(1,1)^3(2,2)}$, $\chi_{(0,0)^2(3,-3)(1,-1)^2}$. In conclusion, the characters with phase -1 $(+1)$ have lowest level states transforming in the antisymmetric (symmetric) representation of $\mathrm{SO}(n_i)$.

Let us work out an example. Consider a particular family of solutions to the tadpole cancellation conditions, namely

$$n_0 = 12-n\,; \qquad n_1 = 20-n\,; \qquad n_2 = n \tag{7.21}$$

where $n_1 = n_{(1,0,0,0,0)}; n_2 = n_{(1,1,0,0,0)}$. The partition function from the cylinder reads

$$\begin{aligned} Z_C = \frac{1}{2}\Big[&(n_A^2+n_B^2+n_C^2)\chi^A + (n_B^2+n_C^2+2n_Bn_A)\chi^B + \\ &+(n_C^2+2n_Cn_A+2n_Cn_B)\chi^C + (n_C^2+2n_Bn_C)\chi^D\Big] \end{aligned} \tag{7.22}$$

where $\chi^A = \frac{1}{5}(\chi_I)^5$; $\chi^B = \frac{1}{5}(\chi_{II})\chi_I^4$; $\chi^C = \frac{1}{5}(\chi_{II})^2\chi_I^3$; $\chi^D = \frac{1}{5}\chi_I\chi_{II}\chi_I^3$, and thus $n_0 = n_{(0,0,0,0,0)} \equiv n_A; n_1 = n_{(1,0,0,0,0)} \equiv n_B$, $n_2 = n_{(1,1,0,0,0)} \equiv n_C$.

χ^A, χ^B, χ^C and χ^D are the combinations of characters contributing to the Möbius strip. As may be seen from table 14 in appendix B, these contain the following massless characters and their charge conjugate combinations: $(0,0)^5$, $(1,-1)^2\underline{(0,0)^2(3,-3)}$ and $(2,-2)\underline{(0,0)^3(3,-3)}$ plus all possible permutations with $(2,-2)$ in the second position.

8. Modding by discrete symmetries

Each of the blocks defining the internal sector of the theory possesses Z_m phase symmetry [4, 46]. Namely, conformal fields transform under such an action as

$$\Phi_{l,q,\bar l,\bar q} \to e^{2i\pi\gamma\frac{(q+\bar q)}{2m}}\Phi_{l,q,\bar l,\bar q} \tag{8.1}$$

with $\gamma \in Z$. Also, since in many cases the internal sector contains several identical conformal blocks, models should be invariant under the permutation of such blocks. Thus, generically the models posses a $G = \otimes_{a=1}^{r} Z_{m_a} \times \mathcal{P}$ group of symmetries ($\mathcal{P}$ denoting block permutations) and therefore it is worth considering the possibility of dividing them out. Such discrete symmetries have been extensively studied in the context of the $E_8 \times E_8$ heterotic string on Gepner and coset models (see for instance [47]–[50]). Generically they lead to a reduction in the number of generations.

Here we show how such moddings can be implemented in the type-IIB orientifold on Gepner points. The general idea is to obtain expressions for *modded* supersymmetric characters with well defined modular transformation properties. Once this is achieved the closed sector is obtained by just plugging left and right modded characters and the Klein-bottle amplitude can be immediately written down in order to proceed to the construction of the D-brane sector.

8.1 Modding out phase symmetries

Consider, as a simple example, the case of just one block closed partition function. In order to mod out the phase symmetry in (8.1) the constraint $\gamma q = 0\, mod\, m$ should be implemented in the left moving sector (and similarly in the right moving one). This can be achieved by introducing the projector $\frac{1}{M}\sum_{x=0}^{M-1} e^{2i\pi\gamma\frac{q}{m}x}$ in the character, where M is the order of the cyclic group G, *i.e.* the least integer such that $M\gamma = 0 \bmod m$. As usual, such a truncation will generically produce a non modular invariant partition function and G-twisted sectors must be added. Twisted sectors can be included in order to ensure that the *modded* character $\chi^G_{l,q}(\tau)$ transforms as the original one. Namely, by defining

$$\chi^G_{l,q}(\tau) = \frac{1}{M}\sum_{x,y=0}^{M-1} e^{2i\pi\gamma^2 y^2\frac{c\tau}{6}} e^{-2i\pi\frac{\gamma^2 xy}{m}} \chi_{l,q}(\tau, \gamma y\tau + \gamma x) \tag{8.2}$$

we can see, for instance, by using the transformation properties for the characters discussed in appendix A that $\chi^G_{l,q}(-1/\tau) = \sum_{l',q'} S_{l,q;l',q'}\chi^G_{l',q'}(\tau)$.

Notice the similarity with the "supersymmetry projection" ($n \equiv y, \gamma \equiv 1/2, p \equiv x$). Therefore, when attached to the antiholomorphic part $\chi^G_{\bar{l},\bar{q}}$ a modular invariant partition function is recovered.

$\chi^G_{l,q}$ can be rewritten (see (A.3) and (A.5) in appendix A) as

$$\chi^G_{l,q}(\tau) = \frac{1}{M}\sum_{x,y} e^{-2i\pi x\frac{\gamma}{m}(q+\gamma y)}\chi_{l,q+2\gamma y}(\tau)\,. \tag{8.3}$$

These considerations are easily generalizable to products of conformal theories in the internal sector of the string which are generically invariant under a cyclic phase symmetry of the form $\otimes_a Z_{M_a}$.

For the sake of simplicity let us mode out by just one Z_{M_a} symmetry. The projection parameters will now be encoded into an r dimensional vector

$$\vec{\Gamma}^a = ({\gamma_1}^a, {\gamma_2}^a, \ldots, {\gamma_r}^a) \tag{8.4}$$

where M_a is the least integer such that

$$M_a{\gamma_i}^a = 0 \quad mod\,(k_i+2) \tag{8.5}$$

and a labels one of the different, inequivalent, moddings.

The product of characters in the internal sector now reads

$$\chi^G_{\vec{l},\vec{q}}(\tau) = \sum_{x,y} e^{-2i\pi x\frac{\gamma_i}{m}(q_i+\gamma_i y)}\chi_{\vec{l},\vec{q}+2\vec{\gamma}y}(\tau) \tag{8.6}$$

where $\vec{l}$, $\vec{q}$ are r-component vectors with entries l_i, q_i respectively. From here we may obtain the projection conditions on each twisted sector $y = 0, \ldots, M_a$

$$\sum_{i=1}^{r} \frac{1}{m_i} \gamma_i^a (q_i + y\gamma_i^a) \in \mathbb{Z}\,. \tag{8.7}$$

Since $\chi^G_{\vec{l},\vec{q}}$ transforms as the original, non projected, character, it is straightforward to write down the supersymmetrized projected character $\chi^{G,susy}_{\vec{\alpha}}$: we must just replace $\chi^G_{l_i,q_i}$ into expression (A.6) in appendix A.

Projection on integer total charge leads to

$$\sum_{i=1}^{r} \frac{1}{m_i} (q_i + 2y\gamma_i^a) \in \mathbb{Z} \tag{8.8}$$

for each twisted sector y. Thus, from (8.7) we see that supersymmetry imposes a further constraint on ${\gamma_i}^a$, namely

$$\sum_{i=1}^{r} \frac{1}{m_i} \gamma_i^a \in \mathbb{Z}\,. \tag{8.9}$$

(This is the usual $2\beta_0 \cdot \Gamma \in \mathbb{Z}$ condition of [4]).

The full modular invariant closed partition function reads

$$\begin{aligned} \sum_{\vec{\alpha},\vec{\bar{\alpha}}} \mathcal{N}_{\vec{\alpha};\vec{\bar{\alpha}}} \chi^{G,\text{susy}}_{\vec{\alpha}} \chi^{G,\text{susy}\ *}_{\vec{\bar{\alpha}}} &= \sum_{\vec{\alpha},\vec{\bar{\alpha}}} \mathcal{N}_{\vec{\alpha};\vec{\bar{\alpha}}} \frac{1}{M} \sum_{x,y} e^{-2i\pi x \frac{\Gamma}{m}(\vec{q}+\Gamma y)} \times \\ &\quad \times \frac{1}{M} \sum_{\bar{x},\bar{y}} e^{-2i\pi \bar{x} \frac{\Gamma}{m}(\vec{\bar{q}}+\Gamma \bar{y})} \chi^{susy}_{\vec{\alpha}+2y\Gamma} \bar{\chi}^{\text{susy}}_{\vec{\bar{\alpha}}+2\bar{y}\Gamma}\,. \end{aligned} \tag{8.10}$$

The left-right symmetric way in which we managed to express the closed partition function permits to immediately write down the associated projected Klein bottle amplitude:

$$\mathcal{Z}^G_K(it) = \sum_{\vec{\alpha}} \mathcal{K}_{\vec{\alpha}} \chi^{G,susy}_{\vec{\alpha}}(2it) = \frac{1}{M} \sum_{\vec{\alpha}} \mathcal{K}_{\vec{\alpha}} \sum_{x,y} e^{-2i\pi x \frac{\Gamma}{m}(\vec{q}+\Gamma y)} \chi^{susy}_{\vec{\alpha}+2y\Gamma} \tag{8.11}$$

where $\mathcal{K}_{\vec{\alpha}} = \mathcal{N}_{\vec{\alpha}\vec{\alpha}}$, from where we can proceed as above in order to obtain the open string sector.

Notice that moddings with no twisted sectors at all are possible. This is indeed the case when (8.7) has the unique solution $y = 0$. The modding is thus freely acting and leads, essentially, to a reduction in the number of states. This is the case, for instance, mentioned in the 3^5 case above for the modding $\Gamma = (-2, -1, 0, 1, 2)$.

However, the presence of G-twisted sectors generically leads to a set of characters which is different from the ones present in the non projected theory producing, *e.g.*, different tadpole cancellation equations. Therefore we expect both the closed and open string sectors to be sensibly modified by the modding. We consider a simple example in the 1^6 model below.

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8.2 Phase modding 1^6

Inequivalent projections are given by

$$\begin{aligned}\Gamma^1 &= (1,1,1,0,0,0) && (8.12)\\ \Gamma^2 &= (1,1,1,-1,-1,-1)\,. && (8.13)\end{aligned}$$

Since $\Gamma^2 = 0 \; mod\; 3$ here, (8.7) reduces to the requirements

$$\begin{aligned}q_1+q_2+q_3 &= 0\, mod\, 3 && (8.14)\\ q_1+q_2+q_3-q_4-q_5-q_6 &= 0\, mod\, 3 && (8.15)\end{aligned}$$

for Γ^1 and Γ^2 respectively, for all twisted sectors $y = 0,1,2$. Let us concentrate in the first modding in the diagonal case.

The projection $q_1+q_2+q_3 = 0$ implies that the only allowed supersymmetric characters (see tables 10–12 in appendix B) are (no permutations here)[8]

$$\begin{aligned}&(0,0)^6\\ &(1,-1)^3(0,0)^3\\ &(1,1)^3(0,0)^3\\ &(0,0)^3\underline{(0,0)(1,-1)(1,1)}\,. && (8.16)\end{aligned}$$

When the spacetime part is included they lead to a vector, two massless matter, (instead of the original 20 massless states) and 6 massive characters respectively.

More explicitly, let us denote by

$$\begin{aligned}NS_1 &\equiv |s;(0,0,0)^3(1,1,0)^3, Q_{\rm int}=1>\\ NS_2 &\equiv |s;(1,-1,0)^3(0,0,0)^3, Q_{\rm int}=-1>\\ (2,1)R_1 &\equiv |(2,1);(0,1,1)^3(0,-1,-1)^3, Q_{\rm int}=0> && (8.17)\end{aligned}$$

the massless scalars and fermions contained in the $(1,-1)^3(0,0)^3$ supersymmetric character. Similarly for $(1,1)^3(0,0)^3$ we have

$$\begin{aligned}NS_3 &\equiv |s;(1,1,0)^3(0,0,0)^3, Q_{\rm int}=1>\\ NS_4 &\equiv |s;(0,0,0)^3(1,-1,0)^3, Q_{\rm int}=-1>\\ (2,1)R_2 &\equiv |(2,1);(0,-1,-1)^3(0,1,1)^3, Q_{\rm int}=0> && (8.18)\end{aligned}$$

and for the vector $(0,0)^3(0,0)^3$

$$\begin{aligned}V &\equiv |(2,2);(0,0,0)^3(0,0,0)^3, Q_{\rm int}=1>\\ (1,2)R_3 &\equiv |(1,2);(0,1,1)^3(0,1,1)^3, Q_{\rm int}=1>\\ (1,2)R_4 &\equiv |(1,2);(0,-1,-1)^3(0,-1,-1)^3, Q_{\rm int}=-1>\ . && (8.19)\end{aligned}$$

[8]Notice that the permuted character $(0,0)^3(1,-1)^3$ and corresponding twists are also allowed by (8.14). However, when all y twists are considered they lead to equivalent characters and must not be counted twice.

Closed sector. The massless states in the closed sector are obtained by coupling, diagonally, the above states for left and right sectors and by keeping invariant combinations under Ω left-right exchange.

The first two characters lead to $8(\mathbf{2},\mathbf{1})$ (from the 16 fermions R_iNS_j), 13 scalars $13(\mathbf{1},\mathbf{1})$ $(4,6,2,1$ fromNS_iNS_i, NS_iNS_j,R_iR_i and R_iR_j, $i \neq j$, respectively) and one tensor multiplet (from R_1R_2). This is the content of $3H+T$. Interestingly enough the number of tensors plus hypers adds up to 4 instead of 20. This is an indication that the original $K3$ of the unmodded 1^6 theory *became a torus* after dividing by the discrete symmetry.

The third character produces an $N=1$ supergravity multiplet when V$-$V coupling is considered. However couplings of the form $V-(2,1)R_{1(2)}$ are now allowed leading, in particular, to two states of the type $2(3,2)$, signaling the presence of extra gravitini as expected in a torus compactification. Actually, it can be checked that all massless states finally arrange into one $N=2$ supergravity multiplet

$$(3,3)+(1,3)+(3,1)+2(2,3)+2(3,2)+(1,1)+4(2,2)+2(1,2)+2(2,1) \tag{8.20}$$

plus four $N=2$ vector multiplets

$$(2,2)+4(1,1)+2(2,1)+2(1,2)\,. \tag{8.21}$$

Open sector. The modded Klein bottle amplitude contains the supersymmetric characters obtained from (8.16) and can be written as

$$\mathcal{Z}_K^G(it)=\frac{1}{2}\left\{\chi^3_{(0,0)}\left(\chi_{(0,0)}+\chi_{(1,-1)}+\chi_{(1,1)}\right)^3\right\}^{\rm susy}(2it) \tag{8.22}$$

which reads, in transverse channel,

$$\tilde{\mathcal{Z}}_K^G(il)=9\,\frac{1}{2}\{\chi^6_{(0,0)}+\chi^3_{(1,-1)}\chi^3_{(0,0)}+\chi^3_{(1,1)}\chi^3_{(0,0)}++\underline{\{\chi_{(0,0)}\chi_{(1,-1)}\chi_{(1,1)}\}\chi^3_{(0,0)}}\}^{\rm susy}(il)\,. \tag{8.23}$$

The following partition functions in the direct and transverse channels of the cylinder and Möbius strip provide a solution with D-brane gauge group $\mathrm{SO}(n_2)\times\mathrm{U}(n_1)$

$$\begin{aligned}
\mathcal{Z}_C(it) &= \left\{\left(n_1^2+\frac{1}{2}n_2^2\right)\sum_j\chi^3_{(0,0)}X_j+(\frac{1}{2}n_1^2+n_1n_2)\sum_j\underline{\chi_{(0,0)}\chi_{(1,-1)}\chi_{(1,1)}}X_j\right\}^{\rm susy}(it)\\
\tilde{\mathcal{Z}}_C(il) &= \frac{1}{2}\Big\{(2n_1+n_2)^2[\chi^6_{(0,0)}+\chi^3_{(1,-1)}\chi^3_{(0,0)}+\chi^3_{(1,1)}\chi^3_{(0,0)}]+\\
&\quad+(n_1-n_2)^2\underline{\chi_{(0,0)}\chi_{(1,-1)}\chi_{(1,1)}\chi^3_{(0,0)}}\Big\}^{\rm susy}(il)\\
\mathcal{Z}_M(it) &= \sum_j(-1)^{N_{(1,1)}}\{-\frac{1}{2}n_2\hat{\chi}^3_{(0,0)}\hat{X}_j-\frac{1}{2}n_1\underline{\{\hat{\chi}_{(0,0)}\hat{\chi}_{(1,-1)}\chi_{(1,1)}\}}\hat{X}_j\}^{\rm susy}(it)\\
\tilde{\mathcal{Z}}_M(is) &= \frac{1}{2}\{-(2n_1+n_2)[\chi^6_{(0,0)}+\chi^3_{(1,-1)}\chi^3_{(0,0)}+\chi^3_{(1,1)}\chi_{(0,0)}]^3-\\
&\quad-(n_1-n_2)e^{-\frac{i\pi}{3}}\underline{\{\chi_{(0,0)}\chi_{(1,-1)}\chi_{(1,1)}\}[\chi_{(0,0)}]^3}\}^{\rm susy}(il)
\end{aligned}$$

where $\sum_i X_i$ denotes the sum over all possible products of three characters and underlining denotes the sum over permutations.

The tadpole cancellation condition leads in this case to $2n_1 + n_2 = 8$. There are two massless hypermultiplets transforming in adjoint representations of $\mathrm{SO}(n_2)$ and $\mathrm{U}(n_1)$ respectively.

It is interesting to notice that the modding of the diagonal $(\mathbf{1_A})^{\mathbf{6}}$ theory we are considering leads to the same Klein bottle amplitude as the $(\mathbf{1_C})^{\mathbf{3}}(\mathbf{1_A})^{\mathbf{3}}$ coupling. Nevertheless, closed sectors are different.

8.3 Cyclic Permutations

Cyclic permutation symmetries in Gepner and coset heterotic models were studied in [47, 48, 49]. Permutation boundary states were considered recently in [14]. q Let us start, for simplicity, with the piece of the internal sector built up from M (M a prime number) identical conformal blocks.

Following [49] we introduce a formal projection operator P over identical states of the $N = 2$ superconformal algebra (not necessarily primary states), such that P acting on a tensor product of states produces a vanishing result unless all states have equal charges and weights. After dividing by this permutation symmetry the following "character" can be defined

$$\chi_{invar}(\tau) = (P + \frac{1-P}{M})\chi^M(\tau) = \frac{1}{M}\chi^M + \frac{M-1}{M}P\chi^M \tag{8.24}$$

$$= \frac{1}{M}\chi^M(\tau) + \frac{M-1}{M}\chi(M\tau) \tag{8.25}$$

where we indicate with a superscript M that the character contains the product of M identical blocks. Also, $P\chi = \mathrm{Tr}Pe^{2i\pi\tau(L_0 - c/24)}$ formally indicates that the traces must be computed by simultaneously considering the same state in all blocks, such that $P\chi^M(\tau) = \chi(M\tau)$ is the character of just one block but evaluated at $M\tau$. Each of these states is counted once. The term $\frac{(1-P)}{M}$ corresponds to the case when at least one state in a block is different from the others. Since this state could belong to any of the M blocks we must divide by M in order to obtain just one full symmetric state. We see that the invariant, untwisted part, will produce the same result as the original partition function but where the states related by a permutation of the M blocks are counted just once.

Consider the closed partition function. Due to the presence of a *fixed point* contribution $\chi(M\tau)$ this partition function is no longer invariant under modular transformations and P-twisted sectors must be added. By starting with the original modular invariant partition function $Z^M(\tau)$ with M identical blocks we finally obtain

$$Z_{\mathrm{new}}(\tau,\bar\tau) = \frac{Z^M(\tau,\bar\tau)}{M} + \frac{M-1}{M}Z(M\tau, M\bar\tau) + \frac{M-1}{M}\sum_{n=0}^{M-1} Z\left(\frac{\tau+n}{M}, \frac{\bar\tau+n}{M}\right). \tag{8.26}$$

Modular invariance can be checked by noticing that twisted sectors

$$\frac{M-1}{M}\sum_{n=0}^{M-1} Z\left(\frac{\tau+n}{M}, \frac{\bar\tau+n}{M}\right) \tag{8.27}$$

and $\frac{M-1}{M}Z(M\tau, M\bar{\tau})$ are related among themselves to the same expressions evaluated in $-1/\tau$ by combinations of $\mathrm{SL}(2,\mathbb{Z})$ actions on the argument.[9]

Therefore, if fields of the original theory are known, (8.26) allows to compute the spectrum in the modded closed string theory. Recall that, apart from the term $\frac{Z^M(\tau,\bar{\tau})}{M}$ containing the full original partition function, in the other terms M blocks have been replaced by just one. In particular it can be shown [48] that twisted sectors can be interpreted as the partition function of a new $N=2$ superconformal field theory with central charge $\hat{c} = Mc$ (c is the central charge of each one of the identical theories) and Virasoro generators given in terms of those of the original theory by

$$\hat{L}_m = \frac{L_{mM}}{M} + \frac{c(M^2-1)}{24M}\delta_{0,m} \tag{8.29}$$

$$\hat{G}^{\pm}_r = \frac{1}{\sqrt{M}}G^{\pm}_{rM} \tag{8.30}$$

$$\hat{J}_m = J_{mM}\,. \tag{8.31}$$

Similar expressions are valid for the right movers.

Thus, weights and charges of the (twisted) primary states of the new theory are obtained from the original ones as

$$h_{\text{new}} = \frac{h+m}{M} + \frac{c(M^2-1)}{24M} \tag{8.32}$$

$$Q_{\text{new}} = Q \tag{8.33}$$

where m is the level of the descendant field.

The sum over n in (8.27) imposes the constraint $h + m - \bar{h} - \bar{m} = 0 \quad mod\ M$.

In order to construct the partition function for the full D dimensional string, the spacetime sector and the other $r - M$ Gepner blocks must be included, and the characters must be supersymmetrized in the usual way [47, 48, 49]. Namely, the full, non modded partition function reads

$$\mathcal{Z}_T(\tau,\bar{\tau}) = \sum_{\vec{\alpha},\vec{\bar{\alpha}}} \chi^{\text{susy}}_{\vec{\alpha}}(\tau)\mathcal{N}^{\vec{\alpha}\vec{\bar{\alpha}}}\chi^{susy\ *}_{\vec{\bar{\alpha}}}(\bar{\tau}) \tag{8.34}$$

where $\vec{\alpha}$ is a $d+r-M+M$ dimensional vector index and the full character is schematically given by

$$\chi^{\text{susy}}_{\vec{\alpha}}(\tau) = \frac{1}{m}\sum_{n,p}\left[\chi_0(\tau, z_{n,p})\right]^d \prod_{i=d+1}^{d+r-M}\chi_{\alpha_i}(\tau, z_{n,p}) \prod_{j=d+r-M+1}^{d+r}\chi^M_{\alpha_j}(\tau, z_{n,p}) \tag{8.35}$$

[9] More explicitly, by choosing

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1+lm}{M} & m \\ l & M \end{pmatrix} \tag{8.28}$$

with l, m chosen such that $\frac{1+lm}{M} \in \mathbb{Z}$, it is easy to see that $\gamma(\tau', z') = (\frac{a\tau'+b}{c\tau'+d}, \frac{z'}{c\tau'+d})$ and then, for $\tau' = \frac{\tau M}{1-l\tau}$, we have $\gamma(\tau') = \frac{\tau+m}{M}$, as required.

with $z_{n,p} = \frac{n}{2}\tau + \frac{p}{2}$ (and similarly for the right sector). Again, the superscript M in the last character indicates a product of M characters corresponding to primary fields of M identical blocks (encoded in the vector $\vec{\alpha}_M$ denoting the last M entries of $\vec{\alpha}$).

The full modular invariant projected partition function is thus

$$\begin{aligned}\mathcal{Z}^{\rm susy}_{\rm new}(\tau) &= \frac{1}{M}\sum_{n,p}\mathcal{Z}^{st}(\tau,z_{n,p})\mathcal{Z}^{r-M}(\tau,z_{n,p})Z^{M}(\tau,z_{n,p}) \\ &+\frac{M-1}{M}\sum_{n,p}\mathcal{Z}^{st}(\tau,z_{n,p})\mathcal{Z}^{r-M}(\tau,z_{n,p})\mathcal{Z}(M\tau,Mz_{n,p}) \\ &+\frac{M-1}{M}\sum_{n,p}\sum_{m=0}^{M-1}\mathcal{Z}^{st}(\tau,z_{n,p})\mathcal{Z}^{r-M}(\tau,z_{n,p})\mathcal{Z}\left(\frac{\tau+m}{M},z_{n,p}\right). \qquad (8.36)\end{aligned}$$

The first term is nothing but the original partition function divided by M. The second one is the fixed point contribution where M identical blocks are replaced by just one evaluated at $M\tau$. The sum of both terms accounts for the permutation invariant contributions as discussed above. The last term is the twisted sector contribution which, as discussed in (8.31), contains a *new* conformal field theory where conformal weights and charges are given in (8.33).

Notice that fixed point and twisted contributions are built up from $r-M+1$ internal blocks. The index vector $\vec{\alpha}$ must now be replaced by a"collapsed" index $\vec{\alpha}'$ and the modular invariant coupling for such terms is

$$\mathcal{N}^{\vec{\alpha}'\bar{\vec{\alpha}}'} = \prod_{i=1}^{d}\mathcal{N}^{\vec{\alpha}_i\bar{\vec{\alpha}}_i}\prod_{j=d+1}^{d+r-M}\mathcal{N}^{\alpha_j\bar{\alpha}_j}\mathcal{N}^{\alpha_M\bar{\alpha}_M}\,. \qquad (8.37)$$

We present a sample computation below.

From (8.36) we can now immediately obtain the Klein bottle partition function. We must just keep the left-right invariant piece and evaluate it at $2Im\tau$ (see (4.4)). In particular the m dependence in the twisted sector drops out and the factor M in the denominator cancels out. Therefore, the Klein bottle amplitude we are led to is

$$\mathcal{Z}_K=\frac{1}{2}\sum_{\vec{\alpha}}\mathcal{K}_{\vec{\alpha}}\chi^{\rm susy}_{\vec{\alpha}}(2it)=\frac{1}{2M}\sum_{\vec{\alpha}}\mathcal{K}_{\vec{\alpha}}\chi^{\rm susy}_{\vec{\alpha}}+\frac{M-1}{2M}\sum_{\vec{\alpha}'}\mathcal{K}_{\vec{\alpha}'}\chi^{fix}_{\vec{\alpha}'}+\frac{(M-1)}{2}\sum_{\vec{\alpha}'}\mathcal{K}_{\vec{\alpha}'}\chi^{\rm twisted}_{\vec{\alpha}'} \qquad (8.38)$$

where $\chi^{\rm susy}_{\vec{\alpha}}$ is the original supersymmetric character introduced in (8.35),

$$\chi^{fix}_{\vec{\alpha}'} = \frac{1}{m}\sum_{n,p}[\chi_0(\tau,z_{n,p})]^d\prod_{i=d+1}^{d+r-M}\chi_{\alpha_i}(\tau,z_{n,p})\chi_{\alpha_M}(2itM,Mz_{n,p}) \qquad (8.39)$$

and

$$\chi^{\rm twisted}_{\vec{\alpha}'} = \frac{1}{m}\sum_{n,p}[\chi_0(2it,z_{n,p})]^d\prod_{i=d+1}^{d+r-M}\chi_{\alpha_i}(2it,z_{n,p})\chi_{\alpha_M}\left(\frac{2it}{M},z_{n,p}\right) \qquad (8.40)$$

($z_{n,p}$ is defined as above with $\tau\to 2it$). Interestingly enough, a similar result was obtained in [14] in terms of boundary states.

Once the amplitude from the Klein bottle is obtained we can follow the usual procedure to build up the open string sector. It is worth noticing that when the characters are expressed in terms of $l = 1/2t$, by using the modular transformations given in (4.5) in order to obtain the transverse channel amplitude, fixed and twisted characters do exchange, namely

$$\chi^{fix}_{\vec{\alpha}'}(2it) = S_{\vec{\alpha}'\vec{\beta}'}\chi^{\text{twisted}}_{\vec{\beta}'}(il)\,. \tag{8.41}$$

Therefore, the Klein bottle amplitude in the transverse channel reads

$$\tilde{\mathcal{Z}}_K = \frac{1}{2M}\sum_{\vec{\alpha}} S_{\vec{\alpha}\vec{\beta}}\mathcal{K}^{\vec{\alpha}}\chi^{\text{susy}}_{\vec{\beta}} + \frac{(M-1)}{2}\sum_{\vec{\alpha}'} S_{\vec{\alpha}'\vec{\beta}'}\mathcal{K}^{\vec{\beta}'}\chi^{fix}_{\vec{\alpha}'} + \frac{M-1}{2M}\sum_{\vec{\alpha}'} S_{\vec{\alpha}'\vec{\beta}'}\mathcal{K}^{\vec{\beta}'}\chi^{\text{twisted}}_{\vec{\alpha}'} \tag{8.42}$$

Recall that the factors in front are different from (8.38).

The first term will lead to the same tadpole structure as the original, non permuted, theory. We do not expect new tadpole contributions to be generated from the fixed point term. In fact, if such contribution exists it is already contained in the first term. The last piece, instead, will contain new states, involving charges and conformal weights given in (8.32) and (8.33), and could produce new tadpole cancellation conditions. Let us consider some examples.

$M = 3$ permutations in diagonal 1^6 model. Consider permutations of the first three blocks in the diagonal 1^6 theory.

In the original theory 20 massless hypermultiplets encoded in the characters $\underline{(0,0,0)^3(1,1,0)^3}$ (and positive charge states $\underline{(0,0,0)^3(1,-1,0)^3}$) are present. (Recall that underlining denotes all possible permutations of the underlined blocks). When we divide out by permutations of the first three blocks, the untwisted sector contribution requires permuted states to be identified (counted just once), thus we are left with

$$(0,0,0)^3(1,1,0)^3 \tag{8.43}$$

$$(1,1,0)^3(0,0,0)^3 \tag{8.44}$$

$$\{(1,1,0)^2(0,0,0)\}\underline{(0,0,0)^2(1,1,0)} \tag{8.45}$$

$$\{(1,1,0)(0,0,0)^2\}\underline{(0,0,0)(1,1,0)^2} \tag{8.46}$$

(and similarly those with opposite charge) for $1+1+3+3 = 8$ hypermultiplets when coupled to the same states in the right moving sector.

We must also include twisted sector contributions.

The conformal weight of a state is the sum of the spacetime and internal conformal weights. If we are interested in massless matter states we must look for all states with $\Delta_{\text{int}} = 1/2$ where the first three theories are now replaced by the new conformal theory in (8.33). We thus see, from the expression for Δ_{new} in (8.33), that masslessness requires $m = 0$. For such $m = 0$ we obtain that

$$\begin{aligned}(l,q,s) &\to (\Delta, \Delta_{\text{new}}, Q_{\text{new}} = Q) \\ (1,1,0) &\to \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)\end{aligned} \tag{8.47}$$

$$
\begin{aligned}
(1,-1,-2) &\to \left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)\\
(1,-1,0) &\to \left(\frac{1}{6},\frac{1}{6},\frac{1}{3}\right)\\
(1,1,2) &\to \left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)
\end{aligned}
$$

which lead to the massless odd charge combinations

$$
\begin{aligned}
&\{(1,1,0)\}_{\rm new}\,\underline{(1,1,0)(1,1,0)(0,0,0)}\\
&\{(1,-1,-2)\}_{\rm new}\,\underline{(1,1,0)(0,0,0)(0,0,0)}\\
&\{(1,-1,0)\}_{\rm new}\,\underline{(1,-1,0)(1,-1,0)(0,0,0)}\\
&\{(1,1,2)\}_{\rm new}\,\underline{(1,-1,0)(0,0,0)(0,0,0)}
\end{aligned} \tag{8.48}
$$

6 massless states with total charge -1 (and other six with charge 1). Since there is an extra factor $M-1=2$, we have a total of 24 massless twisted states that lead to 12 hypermultiplets when coupled to identical states in the right sector. Untwisted and twisted states sum up to a total of 20 corresponding $K3$ moduli, as expected.

It is worth stressing that care must be taken when considering field identifications in the new theory. Fields that were equivalent in the original theory ($Z(\tau)$) are not in the twisted new $Z(\tau/M)$ theory and this could lead to miscountings if not treated properly. Namely, equivalences (3.6), (3.7) read now

$$
\begin{aligned}
\{(l,q,s)\}_{\rm new} \equiv \{(k-l,q+M(k+2),s+2M)\}_{\rm new} &\equiv \{(l,q+2M(k+2),s\}_{\rm new}\\
&\equiv \{(l,q+,s+4M)\}_{\rm new}
\end{aligned} \tag{8.49}
$$

For $M=3$ these equivalences do not lead to extra states. However, the situation is different for instance when $M=5$ permutations are considered. In fact, in such a case it is easy to see that there are 4 untwisted hypers and that the combinations

$$
\begin{aligned}
&\{(0,0,2)\}_{\rm new}(0,0,0)\\
&\{(1,1,2)\}_{\rm new}(1,-1,0)
\end{aligned} \tag{8.50}
$$

are massless twisted contributions. When coupled diagonally to the right movers we would obtain 8 twisted hypermultiplets (recall the $M-1=4$ factor in front) instead of 18 as expected. However, when the new equivalences above are taken into account we find that, when coupling left and right sectors in a diagonal invariant manner, we actually have

$$\{(1,3,4)\}_{\rm new}(0,0,0) \;\;{-}{-}\;\; \{(1,3,4)\}_{\rm new}(0,0,0) \tag{8.51}$$

$$\{(1,1,2)\}_{\rm new}(1,1,0) \;\;{-}{-}\;\; \{(1,1,2)\}_{\rm new}(1,1,0) \tag{8.52}$$

$$\{(1,3,4)\}_{\rm new}(0,0,0) \;\;{-}{-}\;\; \{(1,13,10)\}_{\rm new}[(1,1,0)\equiv(0,2,2)] \tag{8.53}$$

$$\{(1,1,2)\}_{\rm new}(1,1,0) \;\;{-}{-}\;\; \{(1,21,4)\}_{\rm new}[(0,0,0)\equiv(1,3,2)] \tag{8.54}$$

leading to 16 hypermultiplets as required.

Open sector. Let us sketch the construction of the open sector.

The amplitude from the Klein bottle in the transverse sector is generically given in (8.42).

In our 1^6 example the first factor is just $1/M$ times the original one, as given in (6.1). The "twisted" contribution (originated in the direct channel *fixed* point) is proportional to

$$\frac{M-1}{2M}\sum_{n,p}\frac{(-1)^{n+p}}{2m}\left(\chi_{(0,0);p,-n}(il)\right)^3\left(\chi_{(0,0);p,-n}\left(\frac{il}{M}\right)\right). \tag{8.55}$$

Both terms will contribute to tadpoles. A simple solution, as we did in section 6, is to propose a similar partition function for the amplitudes from the cylinder and Möbius strip. Namely, $\tilde{\mathcal{Z}}_C = n^2 = \tilde{\mathcal{Z}}_K$ and $\tilde{\mathcal{Z}}_M = -2n\tilde{\mathcal{Z}}_K(il+1/2)$. Thus, factorization is ensured and tadpole cancellation requires $n = 8$ as before. Therefore, from the direct channel amplitudes we find again an SO(8) gauge group. Now the untwisted sector contributes with 4 (2 in $M = 5$) hypermultiplets in **28** while the twisted sector generates the extra 6 (8) (recall identifications in (8.49)), in order to complete a total of 10 hypermultiplets.

Thus, as advanced, we recover the same massless spectrum where part of it comes from the invariant piece of the partition function while the rest originates in the twisted sector contributions. In fact, this is expected by anomaly cancellation in $D = 6$. Even if **28** is gauge anomaly free, ten such hypers are needed in order to ensure absence of gravitational anomalies.

A similar computation for the 1^9 model in $D = 4$, which originally has 84 (left) states in the **6** of SO(4), leads, for instance, to 12 and 6 untwisted and twisted states, respectively, in the $M = 7$ permutations case, leading to an effective reduction of the number of states. Recall that (this is a general result) due to the term $\frac{c(M^2-1)}{24M}$ in (8.32), there will not be direct contributions from the new twisted sector to vector characters. In principle it could contribute, indirectly, through tadpole cancellation.

As a last example let us consider the 3^5 quintic diagonal $D = 4$ model where permutation of all 5 theories is considered. The twisted sector corresponds now to just one theory with charges and weights given in (8.32), (8.33). From (8.32) we notice that $h_{\text{new}} = \frac{3}{5} + \frac{1}{5}(h+m)$ where h are the conformal weights given in tables 10–12. We thus see that there are no massless twisted states allowed. Moreover, we see that odd total charge condition can not be fulfilled and therefore there is no twisted sector at all. Thus, 5 cyclic permutation twists act freely in this model. The original 101 (charge 1) massless states reduce in this case to 21.

9. Non supersymmetric models

It is interesting to extend previous results in order to incorporate anti-D branes [51, 52, 53, 54, 55]. While supersymmetry will be preserved in the closed sector it will be generically broken in the open sector. Antibranes differ from branes in that they carry opposite RR charges. Namely, when antibranes are incorporated we should have, besides the D-brane RR charges $D_{\vec{\alpha}} = D_{\vec{\alpha}\vec{\alpha}'}n_{\vec{\alpha}'}$ (see (2.6)), antibrane charges terms of the form $-\bar{D}_{\vec{\alpha}} = -D_{\vec{\alpha}\vec{\alpha}'}w_{\vec{\alpha}'}$ where $w_{\vec{\alpha}'}$ is the number of antibranes of "type" $\vec{\alpha}'$ on which open strings can end. Notice that geometrical (conformal theory) terms $D_{\vec{\alpha}}$ are the same for both branes and antibranes since they only differ in the sign of the RR charge.

A first consequence of such inclusion is that tadpole cancellation equations (2.7) are modified and they now read

$$O_{\vec{\alpha}} + D_{\vec{\alpha}} - \bar{D}_{\vec{\alpha}} = 0 . \tag{9.1}$$

for characters containing RR massless fields.

In terms of the partition function, antibranes are thus treated as branes (we must just replace $n_{\vec{\alpha}} \to w_{\vec{\alpha}}$ everywhere) but taking into account that they have opposite signs in RR transverse channel, namely, in characters containing $\vartheta\left[\begin{smallmatrix}\frac{\nu}{2}\\0\end{smallmatrix}\right]$ in the spacetime sector. Therefore, it is easier to first look at the transverse channel and then study changes in the spectrum by transforming to the open string direct channel.

Consider, for instance, the transverse cylinder amplitude originated in direct channel amplitudes containing bosonic states. It must be of the form

$$(D_{\vec{\alpha}} + (-1)^p \bar{D}_{\vec{\alpha}})^2 \chi_{\vec{\alpha}} \left(-\frac{1}{\tau}, \frac{p}{2\tau}\right) = \left[D_{\vec{\alpha}\vec{\beta}}(n_{\vec{\beta}} + (-1)^p w_{\vec{\beta}})\right]^2 \chi_{\vec{\alpha}} \left(-\frac{1}{\tau}, \frac{p}{2\tau}\right) \tag{9.2}$$

where p odd (even) corresponds to RR ($NSNS$) closed bosonic states. When transformed to the direct channel, it reads

$$\left[C^{\vec{\alpha}}_{\vec{\alpha}'\vec{\alpha}''}(n^{\vec{\alpha}'} + (-1)^p w^{\vec{\alpha}'})(n^{\vec{\alpha}''} + (-1)^p w^{\vec{\alpha}''})\right] \chi_{\vec{\alpha}} \left(\tau, \frac{p}{2}\right) . \tag{9.3}$$

Here $\chi_{\vec{\alpha}}$ is the product of characters denoted $\chi'_{\vec{\alpha}}$ in equation (3.33).

Also, since under the P transformation $\vartheta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right] \to \vartheta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]$, the direct channel amplitude from the Möbius strip is

$$\left[M_{\vec{\alpha}\vec{\alpha}'}(n^{\vec{\alpha}'} + w^{\vec{\alpha}'})\right] \chi_{\vec{\alpha}} \left(\tau, \frac{p}{2}\right) . \tag{9.4}$$

Thus, we observe that strings stretching between anti branes produce the same spectrum as in the brane-brane sector. However, when a string stretches between a brane and an antibrane a factor $(-1)^p \chi_{\vec{\alpha}}(\tau, \frac{p}{2})$ does appear. Interestingly enough, when the sum over p is performed, *even* charge states instead of *odd* ones, as required by supersymmetry, are now selected. Therefore, as expected, supersymmetry is broken, in the open sector, by the presence of antibranes. Moreover, even charge will now allow, in particular, (real) scalar tachyons charged under both branes and antibranes gauge groups ($C^{\vec{\alpha}}_{\vec{\alpha}'\vec{\alpha}''} n^{\vec{\alpha}'} w^{\vec{\alpha}''}$).

Similar reasoning leads us to fermionic amplitudes from the cylinder and Möbius strip

$$C^{\vec{\alpha}}_{\vec{\alpha}'\vec{\alpha}''}(n^{\vec{\alpha}'} + w^{\vec{\alpha}'})(n^{\vec{\alpha}''} + w^{\vec{\alpha}''}) \chi_{\vec{\alpha}} \left(\tau, \frac{\tau}{2} + \frac{p}{2}\right) \tag{9.5}$$

$$M^{\vec{\alpha}}_{\vec{\alpha}'}(n^{\vec{\alpha}'} - w^{\vec{\alpha}'}) \chi_{\vec{\alpha}} \left(\tau, \frac{\tau}{2} + \frac{p}{2}\right) . \tag{9.6}$$

In particular we observe that, due to the minus sign in the $O - \bar{D}$ Möbius strip sector antisymmetric $\boxminus$ representations under the brane group become symmetric $\square\!\square$ and vice versa.

$(1_A)^4(1_C)^2$ **example.** From the discussion above and by using results from section 6 for $D - D$ and $O - D$ sectors we obtain

$$\mathrm{U}(n_1) \times \mathrm{SO}(n_2) \times [\,\mathrm{U}(w_1) \times \mathrm{SO}(w_2)\,] \tag{9.7}$$

DD and $\bar{D}\bar{D}$ gauge groups and the massless spectrum is given by

$$\begin{array}{lcc}
\mathbf{DD} + \mathbf{DO} + \mathbf{OD} & \text{Fermions}_+ & 4(1,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + 4(\mathrm{Adj},1) + 6(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1) + 6(\mathrm{n}_1,\mathrm{n}_2) \\
 & \text{Fermions}_- & (1,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + (\mathrm{Adj},1) \\
 & \text{Compl.scalars} & 4(1,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + 4(\mathrm{Adj},1) + 6(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1) + 6(\mathrm{n}_1,\mathrm{n}_2) \\
\bar{\mathbf{D}}\bar{\mathbf{D}} + \bar{\mathbf{D}}\mathbf{O} + \mathbf{O}\bar{\mathbf{D}} & \text{Fermions}_+ & 4(1,\Box\!\Box) + 4(\Box\!\Box,1) + 6(\Box\!\Box,1) + 6(w_1,w_2) \\
 & \text{Fermions}_- & (\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1) + (1,\mathrm{Adj}) \\
 & \text{Compl.scalars} & 4(1,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + 4(\mathrm{Adj},1) + 6(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1) + 6(\mathrm{w}_1,\mathrm{w}_2) \\
\mathbf{D}\bar{\mathbf{D}} + \bar{\mathbf{D}}\mathbf{D} & \text{Fermions}_+ & (n_1,1;w_1,1) + (1,n_2;1,w_2) \\
 & \text{Fermions}_- & 6(1,n_2;w_1,1) + 6(n_1,1;1,w_2) \\
 & \text{tachyons} & (n_1,1;w_1,1) + (1,n_2;1,w_2).
\end{array} \tag{9.8}$$

It is easy to check that all, gauge and gravitational anomalies (see closed sector above) cancel if the following constraint

$$w_2 - n_2 + 2(w_1 - n_1) + 8 = 0 \tag{9.9}$$

is satisfied. This is the tadpole cancellation condition (9.1), as expected.

We see that, due to lack of supersymmetry, scalar tachyons generically appear, signaling instability of the vacuum (recall that the closed sector is still supersymmetric). However, tachyon free non supersymmetric vacua are still possible by choosing, for instance $w_1 = n_2 = 0$. In this case $\mathrm{U}(n_1) \times \mathrm{SO}(2(n_1 - 4))$ where the first (second) factor comes from brane$-$brane (antibrane$-$antibrane) sector. The rank of the gauge group can be arbitrarily high. However this is just a signal (see [52] for similar examples in the orbifold context) that such vacua should be interpreted as excitations of a stable supersymmetric vacuum obtained when branes and antibranes annihilate to leave $n_1 = 4$ D-branes. In principle, models where not all antibranes could annihilate could exist in this context.

In fact, the quintic model provides us with such possibilities. Just to illustrate this issue let us consider, as in example (7.21) the non-vanishing coefficients n_0, n_1, n_2. Tadpole cancellation conditions (7.15) can be rewritten as

$$n_0 + n_1 + 2n_2 = 32\,, \qquad n_1 - n_0 = 8 \tag{9.10}$$

where the first equation tells us that a total of 32 D-branes is needed and the second one plays the role of twisted tadpole cancellation equation [52]. Notice that, while a supersymmetric model with $n_0 - n_1 < 8$ is certainly not allowed, a consistent non-supersymmetric model can be built up. Namely, let us choose $n_0 = 0, n_1 = 4$ and introduce a set of w_0 antibranes of type 0. Tadpole equations (9.1) now read

$$-w_0 + 4 + 2n_2 = 32\,; \qquad w_0 = 4 \tag{9.11}$$

which allow for a consistent, non-supersymmetric model with $\mathrm{SO}(16) \times \mathrm{SO}(4)$ brane-brane group and $\mathrm{SO}(4)$ antibrane-antibrane gauge group .

Moreover, since branes and antibranes are of different kind, no tachyons will be present.

10. Summary and outlook

In this work we have addressed the construction of type-II B orientifolds, in $D = 8, 6$ and $D = 4$ dimensions, where the internal sector is built up from Gepner models by extending some preliminary work on the subject. An important step in this construction is the identification, following original Gepner ideas, of the $N = 1$ *supersymmetric* character $\chi^{\rm susy}_{\vec{\alpha}}(\tau)$ in (3.35) for each moving sector. In particular, once such characters are obtained, it proves rather easy to, formally, implement moddings by phase or permutation symmetries as presented in sections 7 and 8.

Of course, a serious limitation for computing explicit cases is that a big number of characters must be taken into account. This number generically increases with the number of internal dimensions and with the level k of the internal theory. In fact, even if RR tadpole cancellation requires to look only at massless states in the transverse channel, factorization must be checked for all massless and massive states. In some models this requires to consider thousands of characters, a hard task even for a fast computer. Thus, special attention must be dedicated to computational techniques reducing the number of characters to handle. For instance, as we explicitly showed in $D = 8$ examples, computations are strongly simplified by the use of *reduced* modular transformation matrices taking into account only the *odd* charged states, instead of the direct product of block transformation matrices. The use of a full conformal algebra character (given in terms of (l, q) instead of (l, q, s)), also seems to present some advantages in concrete computations of factorization and tadpole cancellation, by reducing the number of states to deal with.

Moreover, besides its potential phenomenological interest, modding by discrete symmetries allows to reduce these numbers. For instance, the hundred massless (left) matter characters in the 3^5 model could be reduced to 4 by simultaneously modding by phase and permutation symmetries. In addition to eight dimensional models which were discussed rather exhaustively to be compared with other constructions, explicit models in $D = 6, 4$ were mainly presented as examples. They illustrate how the number of generations can be modified, how gauge symmetries are enhanced, how phase moddings could induce a topology change, etc.

We hope that this may help to offer guidelines to handle specific models with a phenomenological or theoretical interest. For instance, it seems interesting to reconsider cases like the 3^5 quintic. In our example (see (7.5)) we have chosen to achieve factorization by selecting the same characters in the transverse cylinder amplitudes than those appearing in the Klein bottle contribution. This allowed us to easily show how high rank solutions can be obtained. However, with such choice we also restricted ourselves to symplectic and/or orthogonal groups since corresponding Möbius strip amplitudes are needed in order to complete squares. A possible extension is to look for a generalization where, besides the already considered terms, other characters, not present in the Klein bottle partition function are included in the cylinder amplitude which would thus lead to unitary groups with presumably chiral matter content.

Open string version of 3-generations like heterotic Gepner model [64], which involves both modding by phases and cyclic permutation symmetries, could be interesting to study along the lines of our work.

Also, it would be nice to find realizations of non supersymmetric models with antibranes where complete brane-antibrane annihilation, leading to stable supersymmetric vacuum is not allowed, for instance, by tadpole cancellation conditions, as it is the case in some orbifold compactifications [56].

Extentions of Gepner models including non-diagonal modular invariant couplings [57] or more general coset models, like Kazama Suzuki constructions [58] should be possible to address along the above lines. We hope that our work, based on a partition function approach, could help in clarifying connections with more geometrical interpretations.

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A. Useful properties of the characters

A.1 Characters of $N = 2$ superconformal minimal models and $N = 2$ strings

In this appendix we collect several properties of the characters which are useful to work out various assertions contained in the main body of the article.

Recall the definition of $\chi_{(l,q)}$ in (3.15)

$$\chi_{l,q}(\tau, z) \equiv \mathrm{Tr}_{\mathcal{H}_{(l,q)}}(e^{2\pi i(L_0 - \frac{c}{24})\tau} e^{2\pi i J_0 z}) = \chi_{(l,q,s)}(\tau, z) + \chi_{(l,q,s+2)}(\tau, z)\,. \tag{A.1}$$

Explicit expressions for these characters of the N=2 superconformal minimal models have been computed in references [59, 60, 61] and extensively used in [4, 62]. In terms of Riemann ϑ-functions they read [63]

$$\chi_{l,q}(\tau, z) = \frac{\vartheta\left[{0 \atop 0}\right](\tau, z)\,\vartheta\left[{-\frac{l+1}{m}+\frac{1}{2} \atop \frac{1}{2}}\right](m\tau, 0)\,\eta^3(m\tau)\,e^{\pi i(\frac{l+1}{m}-\frac{1}{2})}}{\eta^3(\tau)\,\vartheta\left[{\frac{l+q+1-m}{2m} \atop 0}\right](m\tau, z)\,\vartheta\left[{\frac{-l+q-1+m}{2m} \atop 0}\right](m\tau, z)} \tag{A.2}$$

where $m = k + 2$, η is the Dedekind function and the definitions and properties of the ϑ-functions can be found in reference [18].

Consider the combinations $\chi^{\pm}_{l,q}(\tau, z) \equiv \chi_{l,q,s}(\tau, z) \pm \chi_{l,q,s+2}(\tau, z)$ with $s = 0, -1$ for NS, R respectively. The following relations may be easily seen from the definition

$$\begin{aligned} \chi^{\pm}_{l,q}(\tau, z + \frac{1}{2}) &= e^{\pi i Q_{l,q}} \chi^{\mp}_{l,q}(\tau, z) \\ \chi^{\pm}_{l,q}(\tau, z + A) &= e^{2\pi i Q_{l,q} A} \chi^{\pm}_{l,q}(\tau, z) \quad A \in Z\,. \end{aligned} \tag{A.3}$$

Using that $mQ_{l,q}$ is an integer number, these properties allow to implement the GSO projection on the product of characters of N=2 strings as

$$\sum_{p=0}^{2m-1} \frac{(-1)^p}{2m} \left[\chi_\nu\left(\tau, z+\frac{p}{2}\right)\right]^d \prod_{i=1}^{r} \chi_{l_i,q_i}\left(\tau, z+\frac{p}{2}\right) = \frac{\delta_{Q_{\vec{\alpha}},Z}}{2}(1-e^{\pi i Q_{\vec{\alpha}}}) \times \\ \times [\chi_\nu(\tau,z)]^d \prod_{i=1}^{r} \chi_{l_i,q_i}(\tau,z)\} \,. \quad \text{(A.4)}$$

It is obvious from this expression that the combinations of characters surviving the GSO projection are those verifying that $\sum_{i=1}^{r} Q_{l_i,q_i,s_i} + \sum_{j=1}^{d} Q_j$ is an odd integer number.

It is also convenient to express the twisted characters $\chi_{l,q+n}(\tau,z)$ in terms of $\chi_{l,q}(\tau,z)$. This can be done by shifting z as follows

$$\chi_{l,q+n}(\tau,z) = e^{2\pi i(n^2\tau\frac{c}{24}+n\frac{c}{6}z)}\chi_{l,q}\left(\tau, z+\frac{n}{2}\tau\right). \quad \text{(A.5)}$$

As mentioned in the text, twisting by $n = 2(k+2)$ one recovers the original character at $n = 0$, except for $l = k/2$ when k is even. In this case, a round trip requires $n = (k+2)$.

Finally, supersymmetry and GSO projection can both be implemented as

$$\chi^{\rm susy}_{\vec{\alpha}}(\tau,z) = \sum_{n,p \bmod 2m} \frac{(-1)^{n+p}}{2m} e^{2\pi i(n^2\tau\frac{c}{24}+n\frac{c}{6}z)} \times \\ \times \left[\chi_0\left(\tau, z+\frac{n}{2}\tau+\frac{p}{2}\right)\right]^d \prod_{i=1}^{r} \chi_{l_i,q_i}\left(\tau, z+\frac{n}{2}\tau+\frac{p}{2}\right) \quad \text{(A.6)}$$

with $c = 12$.

These supersymmetric characters can be split as

$$\chi^{\rm susy}_{\vec{\alpha}}(\tau,z) = \chi^{NS}_{\vec{\alpha}}(\tau,z) - \chi^{R}_{\vec{\alpha}}(\tau,z)\,. \quad \text{(A.7)}$$

where

$$\chi^{NS}_{\vec{\alpha}} = \sum_{even\ n=0}^{2m-2} \chi_{\vec{\alpha}^{(n)}}\,, \qquad \chi^{R}_{\vec{\alpha}} = \sum_{odd\ n=1}^{2m-1} \chi_{\vec{\alpha}^{(n)}}\,. \quad \text{(A.8)}$$

These two blocks contain states with identical charges and conformal weights, therefore (A.7) implies $\chi^{\rm susy}_{\vec{\alpha}}(\tau,z) \equiv 0$.

An alternative decomposition of the supersymmetric characters is the following

$$\chi^{\rm susy}_{\vec{\alpha}}(\tau,z) = \frac{1}{2}((\chi^{+}_{\vec{\alpha}}(\tau,z) - \chi^{-}_{\vec{\alpha}}(\tau,z)) \quad \text{(A.9)}$$

where

$$\begin{aligned}
\chi^{+}_{\vec{\alpha}}(\tau,z) &\equiv \chi^{NS^+}_{\vec{\alpha}}(\tau,z) - \chi^{R^+}_{\vec{\alpha}}(\tau,z) \\
\chi^{NS^+}_{\vec{\alpha}}(\tau,z) &= \sum_{n=0\ (even)}^{2m-2} \delta_{Q_{\vec{\alpha}},Z}\, [\chi^{NS^+}_{\nu+n}(\tau,z)]^d \prod_{i=1}^{r} \chi^{NS^+}_{\alpha_i+n}(\tau,z)\,. \\
\chi^{R^+}_{\vec{\alpha}}(\tau,z) &= \sum_{n=1\ (odd)}^{2m-1} \delta_{Q_{\vec{\alpha}},Z}\, [\chi^{NS^+}_{\nu+n}(\tau,z)]^d \prod_{i=1}^{r} \chi^{NS^+}_{\alpha_i+n}(\tau,z)\,.
\end{aligned} \quad \text{(A.10)}$$

and

$$\begin{aligned}
\chi^{-}_{\vec\alpha}(\tau,z) &\equiv \chi^{NS^-}_{\vec\alpha}(\tau,z) - \chi^{R^-}_{\vec\alpha}(\tau,z)\\
\chi^{NS^-}_{\vec\alpha}(\tau,z) &= \sum_{n=0\ (even)}^{2m-2} \delta_{Q_{\vec\alpha},Z}\, e^{i\pi Q_{\vec\alpha(n)}} \left[\chi^{NS^-}_{\nu+n}(\tau,z)\right]^d \prod_{i=1}^{r} \chi^{NS^-}_{\alpha_i+n}(\tau,z)\\
\chi^{R^-}_{\vec\alpha}(\tau,z) &= \sum_{n=1\ (odd)}^{2m-1} \delta_{Q_{\vec\alpha},Z} \left[\chi^{NS^-}_{\nu+n}(\tau,z)\right]^d \prod_{i=1}^{r} \chi^{NS^-}_{\alpha_i+n}(\tau,z)\,. \qquad \text{(A.11)}
\end{aligned}$$

A.2 Modular transformations of supersymmetric characters

In order to study the modular transformations of $\chi^{\text{susy}}_{\vec\alpha}$ it is convenient to introduce the following notation

$$\chi_{l,q;n,p}(\tau,z) \equiv e^{2\pi i(n^2\frac{c}{24}+n\frac{c}{6}(z+\frac{p}{2}))}\chi_{l,q}\left(\tau, z+\frac{n}{2}\tau+\frac{p}{2}\right) \qquad \text{(A.12)}$$

$$\chi_{\nu;n,p}(\tau,z) = e^{2\pi i(n^2\frac{c_{st}}{24}+n\frac{c_{s-t}}{6}(z+\frac{p}{2}))}\chi_{\nu}\left(\tau, z+\frac{n}{2}\tau+\frac{p}{2}\right). \qquad \text{(A.13)}$$

From (3.23) it follows that

$$\chi_{l,q;n,p}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{2\pi i\frac{cnp}{12}}e^{2\pi i\frac{z^2c}{6\tau}}\sum_{l',q'} S_{l,q;l',q'}\chi_{l',q';p,-n}(\tau,z) \qquad \text{(A.14)}$$

$$\chi_{l,q;n,p}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{2\pi i\frac{cnp}{12}}e^{2\pi i\frac{z^2c}{6\tau}}\sum_{l',q'} S^{-1}_{l,q;l',q'}\chi_{l',q';-p,n}(\tau,-z)\,. \qquad \text{(A.15)}$$

The S modular transformation of $\chi^{\text{susy}}_{\vec\alpha}$ may be obtained multiplying the $S_{l,q;l',q'}$ matrix elements of each individual theory. For the spacetime characters S is the unit matrix with an extra factor $(-i\tau)^d$.

Notice that S exchanges n and p and thus S: $\chi^{NS^+} \longleftrightarrow \chi^{NS^+}, \chi^{NS^-} \longleftrightarrow \chi^{R^+}$ and $\chi^{R^-} \longleftrightarrow \chi^{R^-}$. Moreover, modular invariance implies that it is not possible to achieve supersymmetry without GSO projection and vice versa.

The S modular transformation on the product of characters is as follows

$$\chi^{NS/R}_{\vec\alpha}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = (-i\tau)^{-d}e^{2\pi i\frac{z^2c}{6\tau}}\sum_{\vec\alpha'}\left[\prod_{i=1}^{r} S_{(i)}\right]_{\vec\alpha\vec\alpha'}\chi^{+/-}_{\vec\alpha'}(\tau,z) \qquad \text{(A.16)}$$

$$\chi^{+/-}_{\vec\alpha}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = (-i\tau)^{-d}e^{2\pi i\frac{z^2c}{6\tau}}\sum_{\vec\alpha'}\left[\prod_{i=1}^{r} S_{(i)}\right]_{\vec\alpha\vec\alpha'}\chi^{NS/R}_{\vec\alpha'}(\tau,z)\,. \qquad \text{(A.17)}$$

The sum over $\vec\alpha'$ runs over all vectors with components $\alpha_i=(l_i,q_i)$ in the standard range. Recalling the identity (3.40) and noticing that $\left[\prod_{i=1}^r S_{(i)}\right]_{\vec\alpha\vec\alpha'(n)} = \left[\prod_{i=1}^r S_{(i)}\right]_{\vec\alpha\vec\alpha'}$, for all $\vec\alpha$ such that $Q_{\vec\alpha}$ is an integer number, a matrix S may be defined to act on the independent characters $\chi^{\text{susy}}_{\vec\alpha}$ as follows

$$S_{\vec\alpha\vec\delta} = \frac{m}{2^{\epsilon_{\vec\delta}}}\left[\prod_{i=1}^{r} S_{(i)}\right]_{\vec\alpha\vec\delta} \qquad \text{(A.18)}$$

where $\epsilon_{\vec\delta}=1(0)$ if k_i is even for all i and $\vec\delta$ is short (otherwise) and $\epsilon_{\vec\delta}=0$ if k_i is odd for all i. The rank of S is given by the number of independent supersymmetric characters.

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This S matrix is not symmetric when there is a short vector. Applying it twice one obtains

$$\chi_{\vec{\alpha}}^{NS/R}(\tau, z) = S^2_{\vec{\alpha}\vec{\alpha}'}\chi_{\vec{\alpha}'}^{NS/R}(\tau, -z)\,, \tag{A.19}$$

which, together with (3.19) implies $S^2 = C$, C being the charge conjugation matrix.

Regarding the T transformation one can show that

$$\chi_{\vec{\alpha}}^{NS^{\pm}}(\tau+1, z) = e^{2\pi i(\Delta_{\vec{\alpha}} - \frac{c}{24} - \frac{Q_{\vec{\alpha}}}{2})}\chi_{\vec{\alpha}}^{NS^{\mp}}(\tau, z)\,; \qquad \chi_{\vec{\alpha}}^{R^{\pm}}(\tau+1, z) = e^{2\pi i(\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})}\chi_{\vec{\alpha}}^{R^{\pm}}(\tau, z) \tag{A.20}$$

and

$$\chi_{\vec{\alpha}}^{NS/R}(\tau+1, z) = e^{2\pi i(\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})}\chi_{\vec{\alpha}}^{NS/R}(\tau, z)\,. \tag{A.21}$$

One may think of the phase $e^{2\pi i(\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})}$ as the diagonal element of a matrix

$$T_{\vec{\alpha}} \equiv e^{2\pi i(\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})}\delta_{\vec{\alpha}\vec{\alpha}'}\,. \tag{A.22}$$

Note that the transformation $\mathsf{T}^{(2)} : \tau \to \tau + 2$ can be realized by T^2 as

$$\begin{aligned} \chi_{\vec{\alpha}}^{NS^{+}/NS^{-}}(\tau+2, z) &= e^{4\pi i\Delta_{\vec{\alpha}}}\chi_{\vec{\alpha}}^{NS^{+}/NS^{-}}(\tau, z) \\ \chi_{\vec{\alpha}}^{R^{+}/R^{-}}(\tau+2, z) &= e^{4\pi i\Delta_{\vec{\alpha}}}\chi_{\vec{\alpha}}^{R^{+}/R^{-}}(\tau, z)\,. \end{aligned} \tag{A.23}$$

The diagonal elements are the phases $e^{4\pi i(\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})}$, which reduce to $e^{4\pi i\Delta_{\vec{\alpha}}}$ when acting on non vanishing characters (*i.e.* those with integer $Q_{\vec{\alpha}}$).

A.3 The P transformation

The characters in the direct and transverse channels of the Möbius strip are related by the transformation P: $it + \frac{1}{2} \to \frac{i}{4t} + \frac{1}{2}$. This can be generated from the modular transformations S and T as

$$\mathsf{P} = \mathsf{T}\mathsf{S}\mathsf{T}^2\mathsf{S} \tag{A.24}$$

and it squares to the identity, similarly as the S transformation, namely

$$\mathsf{P}^2 = \mathsf{S}^2 = 1\,. \tag{A.25}$$

There exists a matrix P which performs this transformation on the characters $\chi_{\vec{\alpha}}^{R/NS}$. In terms of the S and T matrices, P reads

$$P = TST^2S^{-1}\,, \tag{A.26}$$

and it can be shown that

$$P = TST^2S^{-1} = TS^{-1}T^2S\,. \tag{A.27}$$

This matrix relates characters with different arguments as

$$\chi_{\vec{\alpha}}^{NS/R}\left(\frac{\tau-1}{2\tau-1}, -\frac{z}{2\tau-1}\right) = (1-2\tau)^{-d}e^{2\pi i\frac{z^2c}{3(2\tau-1)}}\sum_{\vec{\alpha}'}P_{\vec{\alpha}\vec{\alpha}'}\chi_{\vec{\alpha}'}^{NS/R}(\tau, z)\,. \tag{A.28}$$

The following action on the characters is easy to see

$$\sum_{\vec{\alpha}',\vec{\alpha}''} P_{\vec{\alpha}\vec{\alpha}''} P_{\vec{\alpha}''\vec{\alpha}'} \chi^{NS/R}_{\vec{\alpha}'}(\tau, z) = \chi^{NS/R}_{\vec{\alpha}}(\tau, -z) \tag{A.29}$$

so that

$$\sum_{\vec{\alpha}'} (P^2)_{\vec{\alpha}\vec{\alpha}'} \chi^{NS/R}_{\vec{\alpha}'}(\tau, z) = \chi^{NS/R}_{\vec{\alpha}}(\tau, -z) \tag{A.30}$$

and from (3.19) one may show that

$$(P^2)_{\vec{\alpha}'\vec{\alpha}} = C_{\vec{\alpha}\vec{\alpha}'} \,. \tag{A.31}$$

The character $\chi_{\vec{\alpha}}(it + \frac{1}{2})$ contains an expansion in powers of $q \equiv e^{-2\pi t}$ multiplied by a phase $e^{\pi i(\Delta^{GSO}_{\vec{\beta}} - \frac{1}{2})}$. Recalling (3.39), this phase is equal to $\pm e^{\pi i(\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})}$ which squares to $T_{\vec{\alpha}}$, thus we denote it $T^{(\frac{1}{2})}_{\vec{\alpha}}$. Extracting this phase the character becomes real. It is convenient to work in the basis of real characters $\hat{\chi}_{\vec{\alpha}}$ defined as

$$\hat{\chi}^{R/NS}_{\vec{\alpha}^{(n)}}(it + \frac{1}{2}, 0) \equiv e^{-\pi i(\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})} \chi^{R/NS}_{\vec{\alpha}^{(n)}}(t) \,. \tag{A.32}$$

The $\hat{\mathsf{P}}$ transformation connecting direct and transverse real Möbius amplitudes is now performed by the matrix

$$\hat{P} = T^{(-1/2)} S T^2 S^{-1} T^{(1/2)} \tag{A.33}$$

where

$$T^{(1/2)}_{\vec{\alpha}\vec{\alpha}} = e^{\pi i(\Delta_{\vec{\alpha}} - \frac{Q_{\vec{\alpha}}}{2})} \,. \tag{A.34}$$

Notice that $T^{(1/2)}_{\vec{\alpha}^{(n)}\vec{\alpha}^{(n)}} = \pm T^{(1/2)}_{\vec{\alpha}\vec{\alpha}}$ and hence the real characters $\hat{\chi}_{\vec{\alpha}}$ change under twisting as $\hat{\chi}_{\vec{\alpha}^{(n)}}(it + \frac{1}{2}) = \pm \hat{\chi}_{\vec{\alpha}}(it + \frac{1}{2})$. Consequently we may choose one $\vec{\alpha}$ and $T^{(1/2)}$ will be the matrix corresponding to that choice.

Therefore the characters in the direct and transverse channels are related as

$$\hat{\chi}^{NS/R}_{\vec{\alpha}} \left(it + \frac{1}{2} \right) = (2it)^d \hat{P}_{\vec{\alpha}\vec{\alpha}'} \hat{\chi}^{NS/R}_{\vec{\alpha}'} \left(\frac{i}{4t} + \frac{1}{2} \right) . \tag{A.35}$$

B. Spectra of some of the models

In this appendix we list the GSO projected combinations of states contained in the characters of some Gepner models.

In the spacetime columns we write the Weyl weights of the spinor and vector representations of the little group. Massive states will gather in representations of the full group. For example, the models in $D = 8$ have SO(6) as little group and SO(7) as full group, and the weights given are those of SO(6) (even for massive states).

B.1 Spectrum of states of 1^3

The GSO projected combinations of states contained in the characters of the 1^3 Gepner model are given in tables 3 and 4.

highest weight state	spacetime	Δ_{st}	Q_{st}	$\Delta_{\rm int}$	$Q_{\rm int}$	Δ	Q
$(0,0,0)^3$	$\underline{(\pm1,0,0)}$	$\frac{1}{2}$	±1	0	0	$\frac{1}{2}$	±1
$(0,1,1)^3$	$\underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$\frac{3}{8}$	$\frac{1}{2}$ $-\frac{3}{2}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	1 -1
$(1,-1,0)^3$	$(0,0,0)$	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1
$(1,1,0)^3$	$(0,0,0)$	0	0	$\frac{1}{2}$	-1	$\frac{1}{2}$	-1
$(0,-1,-1)^3$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $\underline{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$\frac{3}{2}$ $-\frac{1}{2}$	$\frac{1}{8}$	$-\frac{1}{2}$	$\frac{1}{2}$	1 -1

Table 3: GSO projected states in $\chi^{\rm susy}_{(0,0)^3}$.

highest weight state	spacetime	Δ_{st}	Q_{st}	$\Delta_{\rm int}$	$Q_{\rm int}$	Δ	Q
$(0,0,0)(1,-1,2)(1,1,0)$	$(0,0,0)$	0	0	$\frac{5}{6}$	-1	$\frac{5}{6}$	-1
$(0,0,0)(1,-1,0)(1,1,2)$	$(0,0,0)$	0	0	$\frac{5}{6}$	1	$\frac{5}{6}$	1
$(0,0,0)(1,-1,0)(1,1,0)$	$\underline{(\pm1,0,0)}$	$\frac{1}{2}$	±1	$\frac{1}{3}$	0	$\frac{5}{6}$	±1
$(0,1,1)(1,0,-1)(0,-1,-1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $\underline{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$\frac{3}{2}$ $-\frac{1}{2}$	$\frac{11}{24}$	$-\frac{1}{2}$	$\frac{5}{6}$	1 -1
$(0,1,1)(1,0,1)(0,-1,-1)$	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ $\underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$-\frac{3}{2}$ $\frac{1}{2}$	$\frac{11}{24}$	$\frac{1}{2}$	$\frac{5}{6}$	-1 1
$(1,-1,2)(1,1,0)(0,0,0)$	$(0,0,0)$	0	0	$\frac{5}{6}$	-1	$\frac{5}{6}$	-1
$(1,-1,0)(1,1,2)(0,0,0)$	$(0,0,0)$	0	0	$\frac{5}{6}$	1	$\frac{5}{6}$	1
$(1,-1,0)(1,1,0)(0,0,0)$	$\underline{(\pm1,0,0)}$	$\frac{1}{2}$	±1	$\frac{1}{3}$	0	$\frac{5}{6}$	±1
$(1,0,-1)(0,-1,-1)(0,1,1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $\underline{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$\frac{3}{2}$ $-\frac{1}{2}$	$\frac{11}{24}$	$-\frac{1}{2}$	$\frac{5}{6}$	1 -1
$(1,0,1)(0,-1,-1)(0,1,1)$	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ $\underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$-\frac{3}{2}$ $\frac{1}{2}$	$\frac{11}{24}$	$\frac{1}{2}$	$\frac{5}{6}$	-1 1
$(1,1,0)(0,0,0)(1,-1,-2)$	$(0,0,0)$	0	0	$\frac{5}{6}$	-1	$\frac{5}{6}$	-1
$(1,1,2)(0,0,0)(1,-1,0)$	$(0,0,0)$	0	0	$\frac{5}{6}$	1	$\frac{5}{6}$	1
$(1,1,0)(0,0,0)(1,-1,0)$	$\underline{(\pm1,0,0)}$	$\frac{1}{2}$	±1	$\frac{1}{3}$	0	$\frac{5}{6}$	±1
$(0,-1,-1)(0,1,1)(1,0,-1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $\underline{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$\frac{3}{2}$ $-\frac{1}{2}$	$\frac{11}{24}$	$-\frac{1}{2}$	$\frac{5}{6}$	1 -1
$(0,-1,-1)(0,1,1)(1,0,1)$	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ $\underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$-\frac{3}{2}$ $\frac{1}{2}$	$\frac{11}{24}$	$\frac{1}{2}$	$\frac{5}{6}$	-1 1

Table 4: GSO projected states in $\chi^{\rm susy}_{(0,0)(1,-1)(1,1)}$

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Spectrum of states of 2^2. All the representations of the $k = 2$ minimal model are obtained by twisting the pairs $(0,0,0)$; $(0,0,2)$ and $(1,-1,0)$; $(1,-1,2)$, namely

n	Representation	Δ	Q
0	$(0,0,0)$	0	0
1	$(0,1,1)$	$\frac{1}{16}$	$\frac{1}{4}$
2	$(0,2,2) \sim (2,-2,0)$	$\frac{1}{4}$	$\frac{1}{2}$
3	$(2,-1,1)$	$\frac{9}{16}$	$\frac{3}{4}$
4	$(2,0,2)^*$	1	± 1
5	$(2,1,-1)$	$\frac{9}{16}$	$-\frac{3}{4}$
6	$(2,2,0)$	$\frac{1}{4}$	$-\frac{1}{2}$
7	$(2,3,1)$	$\frac{1}{16}$	$-\frac{1}{4}$

n	Representation	Δ	Q
0	$(0,0,2) \sim (2,\pm 4,\pm 4)$	$\frac{3}{2}$	± 1
1	$(0,1,3) \sim (1,-2,-3)$	$\frac{17}{16}$	$-\frac{3}{4}$
2	$(2,-2,-2)$	$\frac{3}{4}$	$-\frac{1}{2}$
3	$(2,-1,-1)$	$\frac{9}{16}$	$-\frac{1}{4}$
4	$(2,0,0)$	$\frac{1}{2}$	0
5	$(2,1,1)$	$\frac{9}{16}$	$\frac{1}{4}$
6	$(2,2,2)$	$\frac{3}{4}$	$\frac{1}{2}$
7	$(2,3,3)$	$\frac{17}{16}$	$\frac{3}{4}$

n	Representation	Δ	Q
0	$(1,-1,0)$	$\frac{1}{8}$	$\frac{1}{4}$
1	$(1,0,1)$	$\frac{5}{16}$	$\frac{1}{2}$
2	$(1,1,2)$	$\frac{5}{8}$	$\frac{3}{4}$
3	$(1,2,3)$	$\frac{17}{16}$	1
4	$(1,-1,-2)$	$\frac{5}{8}$	$-\frac{3}{4}$
5	$(1,0,-1)$	$\frac{5}{16}$	$-\frac{1}{2}$
6	$(1,1,0)$	$\frac{1}{8}$	$-\frac{1}{2}$
7	$(1,2,1)$	$\frac{1}{16}$	0

n	Representation	Δ	Q
0	$(1,-1,-2)$	$\frac{5}{8}$	$-\frac{3}{4}$
1	$(1,0,-1)$	$\frac{5}{16}$	$-\frac{1}{2}$
2	$(1,1,0)$	$\frac{1}{8}$	$-\frac{1}{4}$
3	$(1,2,1)$	$\frac{1}{16}$	0
4	$(1,-1,0)$	$\frac{1}{8}$	$\frac{1}{4}$
5	$(1,0,1)$	$\frac{5}{16}$	$\frac{1}{2}$
6	$(1,1,2)$	$\frac{5}{8}$	$\frac{3}{4}$
7	$(1,2,3)$	$\frac{17}{16}$	1

Table 5: Representations of $k = 2$ minimal model.

From the definition of $\chi^{\rm susy}_{\vec{\alpha}}$ the following identities among characters hold

$$\begin{aligned} \chi^{\rm susy}_{(0,0)^2} &\equiv \chi^{\rm susy}_{(2,-2)^2} \equiv \chi^{\rm susy}_{(2,0)^2} \equiv \chi^{\rm susy}_{(2,2)^2} \\ \chi^{\rm susy}_{(0,0)(2,0)} &\equiv \chi^{\rm susy}_{(2,-2)(2,2)} \equiv \chi^{\rm susy}_{(2,0)(0,0)} \equiv \chi^{\rm susy}_{(2,2)(2,-2)} \\ \chi^{\rm susy}_{(1,-1)(1,1)} &\equiv \chi^{\rm susy}_{(1,1)(1,-1)} \,. \end{aligned} \tag{B.1}$$

The GSO projected combinations of states in the 2^2 Gepner model are given in tables 6, 7 and 8.

Internal Theory	spacetime	Δ_{st}	Q_{st}	Δ_{int}	Q_{int}	Δ	Q
$(0,0,0)^2$	$\underline{(\pm 1,0,0)}$	$\frac{1}{2}$	± 1	0	0	$\frac{1}{2}$	± 1
$(0,1,1)^2$	$\underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$\frac{3}{8}$	$\frac{1}{2}$ $-\frac{3}{2}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	1 -1
$(0,2,2)^2$	$(0,0,0)$	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1
$(2,2,0)^2$	$(0,0,0)$	0	0	$\frac{1}{2}$	-1	$\frac{1}{2}$	-1
$(2,3,1)^2$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $\underline{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$-\frac{1}{2}$ $\frac{3}{2}$	$\frac{1}{8}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1 1

Table 6: GSO projected states in $\chi^{\text{susy}}_{(0,0)^2}$.

Internal Theory	spacetime	Δ_{st}	Q_{st}	Δ_{int}	Q_{int}	Δ	Q
$(0,0,0)(2,0,0)$	$\underline{(\pm 1,0,0)}$	$\frac{1}{2}$	± 1	$\frac{1}{2}$	0	1	± 1
$(0,0,0)(2,0,2)$	$(0,0,0)$	0	0	1	± 1	1	± 1
$(0,1,1)(2,1,1)$	$\underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$\frac{3}{8}$	$\frac{1}{2}$ $-\frac{3}{2}$	$\frac{5}{8}$	$\frac{1}{2}$	1	1 -1
$(0,1,1)(2,1,-1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $\underline{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$-\frac{1}{2}$ $\frac{3}{2}$	$\frac{5}{8}$	$-\frac{1}{2}$	1	-1 1
$(0,2,2)(2,2,0)$	$\underline{(\pm 1,0,0)}$	$\frac{1}{2}$	± 1	$\frac{1}{2}$	0	1	± 1
$(0,2,2)(2,2,2)$	$(0,0,0)$	0	0	1	1	1	1
$(2,-2,-2)(2,2,0)$	$(0,0,0)$	0	0	1	-1	1	-1
$(2,-1,-1)(2,3,1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $\underline{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$-\frac{1}{2}$ $\frac{3}{2}$	$\frac{5}{8}$	$-\frac{1}{2}$	1	-1 1
$(2,-1,1)(2,3,1)$	$\underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$\frac{3}{8}$	$\frac{1}{2}$ $-\frac{3}{2}$	$\frac{5}{8}$	$\frac{1}{2}$	1	1 -1

Table 7: GSO projected states in $\chi^{\text{susy}}_{(1,-1)(1,1)}$.

Internal Theory	spacetime	Δ_{st}	Q_{st}	Δ_{int}	Q_{int}	Δ	Q
$(1,-1,0)(1,1,0)$	$\underline{(\pm 1,0,0)}$	$\frac{1}{2}$	± 1	$\frac{1}{4}$	0	$\frac{3}{4}$	± 1
$(1,-1,2)(1,1,0)$	$(0,0,0)$	0	0	$\frac{3}{4}$	1	$\frac{3}{4}$	1
$(1,-1,0)(1,1,2)$	$(0,0,0)$	0	0	$\frac{3}{4}$	1	$\frac{3}{4}$	1
$(1,0,1)(1,2,1)$	$\underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$\frac{3}{8}$	$\frac{1}{2}$ $-\frac{3}{2}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1 -1
$(1,0,-1)(1,2,1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $\underline{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{3}{8}$	$-\frac{1}{2}$ $\frac{3}{2}$	$\frac{3}{8}$	$-\frac{1}{2}$	$\frac{3}{4}$	-1 1

Table 8: GSO projected states in $\chi^{\text{susy}}_{(0,0);(2,0)}$.

B.2 Spectrum of states of 4 1

The states in the $k = 4$ minimal model are listed in table 9.

n	Representation	Δ	Q
0	$(0,0,0)$	0	0
1	$(0,1,1)$	$\frac{1}{12}$	$\frac{1}{3}$
2	$(0,2,2) \sim (4,-4,0)$	$\frac{1}{3}$	$\frac{2}{3}$
3	$(4,-3,1)$	$\frac{3}{4}$	1
4	$(4,-2,2)$	$\frac{4}{3}$	$\frac{4}{3}$
5	$(4,-1,-1)$	$\frac{13}{12}$	$-\frac{1}{3}$
6	$(4,0,0)$	1	0
7	$(4,1,1)$	$\frac{13}{12}$	$\frac{1}{3}$
8	$(4,2,2)$	$\frac{4}{3}$	$\frac{2}{3}$
9	$(4,3,-1)$	$\frac{3}{4}$	-1
10	$(4,4,0)$	$\frac{1}{3}$	$-\frac{2}{3}$
11	$(4,5,1)$	$\frac{1}{12}$	$-\frac{1}{3}$

n	Representation	Δ	Q
0	$(0,0,2) \sim (4,\pm 6,\pm 4)$	$\frac{3}{2}$	± 1
1	$(4,-5,-3)$	$\frac{13}{12}$	$-\frac{2}{3}$
2	$(4,-4,-2)$	$\frac{5}{6}$	$-\frac{1}{3}$
3	$(4,-3,-1)$	$\frac{3}{4}$	0
4	$(4,-2,0)$	$\frac{5}{6}$	$\frac{1}{3}$
5	$(4,-1,1)$	$\frac{13}{12}$	$\frac{2}{3}$
6	$(4,0,\pm 2)$	$\frac{3}{2}$	± 1
7	$(4,1,-1)$	$\frac{13}{12}$	$-\frac{2}{3}$
8	$(4,2,0)$	$\frac{5}{6}$	$-\frac{1}{3}$
9	$(4,3,1)$	$\frac{3}{4}$	0
10	$(4,4,2)$	$\frac{5}{6}$	$\frac{1}{3}$
11	$(4,5,3)$	$\frac{13}{12}$	$\frac{2}{3}$

n	Representation	Δ	Q
0	$(2,0,0)$	$\frac{1}{3}$	0
1	$(2,1,1)$	$\frac{5}{12}$	$\frac{1}{3}$
2	$(2,2,2)$	$\frac{2}{3}$	$\frac{2}{3}$
3	$(2,\pm 3,\pm 3)$	$\frac{13}{12}$	± 1
4	$(2,-2,-2)$	$\frac{2}{3}$	$-\frac{2}{3}$
5	$(2,-1,-1)$	$\frac{5}{12}$	$-\frac{1}{3}$

n	Representation	Δ	Q
0	$(2,0,\pm 2)$	$\frac{5}{6}$	± 1
1	$(2,1,-1)$	$\frac{5}{12}$	$-\frac{2}{3}$
2	$(2,2,0)$	$\frac{1}{6}$	$-\frac{1}{3}$
3	$(2,3,1)$	$\frac{1}{12}$	0
4	$(2,4,2)$	$\frac{1}{6}$	$\frac{1}{3}$
5	$(2,-1,1)$	$\frac{5}{12}$	$\frac{2}{3}$

Table 9: Representations in $k = 4$ minimal model.

B.3 Spectrum of states of 1^6

Internal Theory	spacetime	Δ_{st}	Q_{st}	Δ_{int}	Q_{int}	Δ	Q
$(0,0,0)^6$	$\underline{(\pm 1,0)}$	$\frac{1}{2}$	± 1	0	0	$\frac{1}{2}$	± 1
$(0,1,1)^6$	$\underline{(-\frac{1}{2},\frac{1}{2})}$	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$\frac{1}{2}$	1
$(0,-1,-1)^6$	$\underline{(-\frac{1}{2},\frac{1}{2})}$	$\frac{1}{4}$	0	$\frac{1}{4}$	-1	$\frac{1}{2}$	-1

Table 10: GSO projected states in $\chi^{\text{susy}}_{(0,0)^6}$.

Internal Theory	spacetime	Δ_{st}	Q_{st}	Δ_{int}	Q_{int}	Δ	Q
$(0,0,0)^4(1,-1,2)(1,1,0)$	$(0,0)$	0	0	$\frac{5}{6}$	-1	$\frac{5}{6}$	-1
$(0,0,0)^4(1,-1,0)(1,1,2)$	$(0,0)$	0	0	$\frac{5}{6}$	1	$\frac{5}{6}$	1
$(0,0,0)^4(1,-1,0)(1,1,0)$	$\underline{(\pm 1,0)}$	$\frac{1}{2}$	± 1	$\frac{1}{3}$	0	$\frac{5}{6}$	± 1
$(1,-1,0)^4(1,1,0)(0,0,0)$	$(0,0)$	0	0	$\frac{5}{6}$	1	$\frac{5}{6}$	1
$(1,1,0)^4(0,0,0)(1,-1,0)$	$(0,0)$	0	0	$\frac{5}{6}$	-1	$\frac{5}{6}$	-1
$(0,1,1)^4(1,0,-1)(1,-1,-1)$	$(\frac{1}{2},\frac{1}{2})$	$\frac{1}{4}$	1	$\frac{7}{12}$	0	$\frac{5}{6}$	1
$(0,1,1)^4(1,0,1)(1,-1,-1)$	$(-\frac{1}{2},-\frac{1}{2})$	$\frac{1}{4}$	-1	$\frac{7}{12}$	0	$\frac{5}{6}$	-1
$(0,1,1)^4(1,0,1)(1,-1,-1)$	$\underline{(\frac{1}{2},-\frac{1}{2})}$	$\frac{1}{4}$	0	$\frac{7}{12}$	1	$\frac{5}{6}$	1
$(0,-1,-1)^4(0,1,1)(1,0,1)$	$(\frac{1}{2},\frac{1}{2})$	$\frac{1}{4}$	1	$\frac{7}{12}$	0	$\frac{5}{6}$	1
$(0,-1,-1)^4(0,1,1)(1,0,1)$	$(-\frac{1}{2},-\frac{1}{2})$	$\frac{1}{4}$	-1	$\frac{7}{12}$	0	$\frac{5}{6}$	-1
$(0,-1,-1)^4(0,1,1)(1,0,-1)$	$\underline{(\frac{1}{2},-\frac{1}{2})}$	$\frac{1}{4}$	0	$\frac{7}{12}$	-1	$\frac{5}{6}$	-1

Table 11: GSO projected states in $\chi^{\text{susy}}_{(0,0)^4(1,-1)(1,1)}$.

Internal Theory	spacetime	Δ_{st}	Q_{st}	Δ_{int}	Q_{int}	Δ	Q
$(0,0,0)^3(1,-1,0)^3$	$(0,0)$	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1
$(1,1,0)^3(0,0,0)^3$	$(0,0)$	0	0	$\frac{1}{2}$	-1	$\frac{1}{2}$	-1
$(0,-1,-1)^3(0,1,1)^3$	$(\frac{1}{2},\frac{1}{2})$ $(-\frac{1}{2},-\frac{1}{2})$	$\frac{1}{4}$	$+1$ -1	$\frac{1}{4}$	0	$\frac{1}{2}$	$+1$ -1

Table 12: GSO projected states in $\chi^{\text{susy}}_{(0,0)^3(1,-1)^3}$.

B.4 Spectrum of states of 3^5

The states in the $k = 3$ model may be classified in 2 groups: those with $l = 0, 3$ and those with $l = 1, 2$. They are

Representation	Δ	Q
$(0,0,0)$	0	0
$(3,-3,0)$	$\frac{3}{10}$	$\frac{3}{5}$
$(3,-1,\pm 2)$	$\frac{6}{5}$	$-\frac{4}{5}, \frac{6}{5}$
$(3,1,0)$	$\frac{7}{10}$	$-\frac{1}{5}$
$(3,3,2)$	$\frac{4}{5}$	$\frac{2}{5}$

Representation	Δ	Q
$(0,0,2)$	$\frac{3}{2}$	± 1
$(3,-3,-2)$	$\frac{4}{5}$	$-\frac{2}{5}$
$(3,-1,0)$	$\frac{7}{10}$	$\frac{1}{5}$
$(3,1,\pm 2)$	$\frac{6}{5}$	$\frac{4}{5}, -\frac{6}{5}$
$(3,3,0)$	$\frac{3}{10}$	$-\frac{3}{5}$

Representation	Δ	Q
$(1,-1,0)$	$\frac{1}{10}$	$\frac{1}{5}$
$(1,1,2)$	$\frac{3}{5}$	$\frac{4}{5}$
$(2,-2,2)$	$\frac{7}{10}$	$\frac{7}{5}$
$(2,0,\pm 2)$	$\frac{9}{10}$	± 1
$(2,2,0)$	$\frac{1}{5}$	$-\frac{2}{5}$

Representation	Δ	Q
$(1,-1,-2)$	$\frac{3}{5}$	$-\frac{4}{5}$
$(1,1,0)$	$\frac{1}{10}$	$-\frac{1}{5}$
$(2,-2,0)$	$\frac{1}{5}$	$\frac{2}{5}$
$(2,0,0)$	$\frac{2}{5}$	0
$(2,2,2)$	$\frac{7}{10}$	$-\frac{7}{5}$

Table 13: Representations of the $k = 3$ minimal model.

where both groups contain also the states obtained by twisting.

It is useful to list the GSO projected combinations with conformal weight $\frac{1}{2}$ in the NS sector, namely

Internal Theory	spacetime	Character
$(0,0,0)^5$	± 1	$\chi_{(0,0)^5}$
$(0,0,0)^3(3,-3,0)(2,-2,0)$	0	$\chi_{(0,0)^3(3,-3)(2,-2)}$
$(0,0,0)^2(3,-3,0)(1,-1,0)^2$	0	$\chi_{(0,0)^2(3,-3)(1,-1)^2}$
$(1,1,0)^5$	0	$\chi_{(2,0)^5}$
$(0,0,0)(1,-1,0)^3(2,-2,0)$	0	$\chi_{(0,0)(1,-1)^3(2,-2)}$
$(0,0,0)^2(1,-1,0)(2,-2,0)^2$	0	$\chi_{(0,0)^2(1,-1)(2,-2)^2}$

Table 14: GSO projected states with conformal weight 1/2.

and their charge conjugated ones.

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